COMPARISON BETWEEN THE FINITE VLBI MODEL AND THE CONSENSUS MODEL

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ABSTRACT. For application of VLBI technique to spacecraft navigation, we have developed an analytical formula of VLBI delay model for finite distance radio source with based on linearized parameterized post Newtonian metric. This formula corresponds to the standard VLBI model ('consensus model'), which is widely used in VLBI community all over the world. This finite VLBI model has an accuracy better than 5 pico seconds for a radio source at distance beyond 10^9 m away from the observer on earth based baseline. The deviation of finite VLBI model from the consensus model comes from the curvature of the wavefront. We compared these two models and derived an analytical expression for the difference.

1 Introduction

Spacecraft navigation is another application field of Very Long Baseline Interferometry (VLBI) other than astronomy and geodesy. VLBI has great sensitivity in direction perpendicular to the line of sight. It is complementary characteristic to the range and range rate (R&RR) measurement, which is widely used for spacecraft navigation in deep space. Thus jointly using VLBI and R&RR together will make increase the accuracy of orbit determination of the spacecraft. An accurate delay model is required in VLBI data processing and analysis, especially for precise astrometry and geodesy. The 'consensus model' [Eubanks, 1991] is widely used as standard VLBI model in world wide VLBI community. That is based on plane wave approximation, however, this standard VLBI model does not have enough accuracy, when radio source is closer than 30 light years from observer (e.g. planets, asteroids, and spacecraft in the solar system). Because curvature of the wave front cannot be ignored in those observations, an alternative VLBI delay model is required for finite distance radio sources.

Sovers & Jacobs (1996) discussed on curvature effect of finite distance radio source. Fukushima (1994) introduced an useful expression of VLBI delay model for finite distance radio source. However, an analytical formula of relativistic VLBI delay model corresponding to the 'Consensus model' was not obviously expressed on their papers. The Jet Propulsion Laboratory (JPL/NASA) have been using VLBI technique for spacecraft navigation [e.g. Border et al., 1982] sometimes. Moyer (2000) has developed formulation of radiometric observation data for spacecraft navigation with based on light time equation (light-time approach). Solving light time equation by numerical procedure is straightforward, however it need iteration of computation to solve light time equation in the analysis software. We took a VLBI-like approach rather than the light-time approach, because we intended to find a analytical formula for replacement with consensus model so that it can easily be implemented in current apriori computation software (CALC[1]). We derived an analytical formula of VLBI delay model for finite distance radio source (hear after referred as 'finite VLBI model') [Sekido and Fukushima, 2003] based of linearized Parameterized Post Newtonian (PPN) metric, and by following the approaches of Hellings (1986), Shahid-Saless and Hellings (1991), and Fukushima (1994). The form is

$$\tau_2 - \tau_1 = (1+\beta_{02})^{-1}$$
$$\left\{\Delta t_g - \frac{\vec{\mathbf{K}}\cdot\vec{\mathbf{b}}}{c}\left[1-(1+\gamma)U - \frac{V_e^2 + 2\vec{\mathbf{V}}_e\cdot\vec{\mathbf{w}}_2}{2c^2}\right] - \frac{\vec{\mathbf{V}}_e\cdot\vec{\mathbf{b}}}{c^2}\left(1+\beta_{02} - \frac{\vec{\mathbf{K}}\cdot(\vec{\mathbf{V}}_e + 2\vec{\mathbf{w}}_2)}{2c}\right)\right\}, \quad (1)$$

where, $\beta_{02} = \hat{\vec{\mathbf{R}}}_{02}\cdot\vec{\mathbf{V}}_2/c$. Target accuracy of this formula is order of pico seconds with ground based baseline for the radio source in the solar system. An expression of the difference between the finite VLBI model and the consensus model is mainly discussed in this paper.

2 Preparation of comparison

2.1 Definitions of Parameters and Notation

Variables of large capital indicate quantity in the rest frame of solar system barycenter (hereafter referred as FCB) and small ones denote those of geocentric reference frame (hereafter referred as FCG). Then Barycentric Coordinate Time (TCB) is represented by T and Geocentric Coordinate Time (TCG) is represented by t. Suffix 0,1,2 represent radio source and two VLBI observation stations, respectively. Position vector of i^{th} station in the FCB is expressed by $\vec{\mathbf{X}}_i$. Radio signal is supposed to be emitted at T_0 from the radio source and to arrive at i^{th} observation station at T_i. Relative position vector and its magnitude are denoted as follows:

$$\vec{\mathbf{R}}_{ij} = \vec{X}_i - \vec{X}_j, \quad R_{ij} = |\vec{\mathbf{R}}_{ij}|$$

Definition of other parameters are summarized in Table 1. Pseudo source vector $\vec{\mathbf{K}}$, baseline vector $\vec{\mathbf{B}}$, and those composed from position vectors of different time epoch are defined in FCB as:

$$\vec{\mathbf{K}} = \frac{\vec{\mathbf{R}}_{01}(T_1) + \vec{\mathbf{R}}_{02}(T_1)}{R_{01}(T_1) + R_{02}(T_1)}, \quad \vec{\mathbf{K}}^* = \frac{\vec{\mathbf{R}}_{01}(T_1) + \vec{\mathbf{R}}_{02}(T_2)}{R_{01}(T_1) + R_{02}(T_2)},$$
$$\vec{\mathbf{B}} = \vec{\mathbf{R}}_{02}(T_1) - \vec{\mathbf{R}}_{01}(T_1), \quad \vec{\mathbf{B}}^* = \vec{\mathbf{R}}_{02}(T_2) - \vec{\mathbf{R}}_{01}(T_1), \quad (2)$$

Table 1: Notation of parameters

$\vec{\mathbf{V}}_e$	Velocity of the earth motion around the sun in FCB
$\vec{\mathbf{w}}_2$	Velocity of the station 2 due to spin of the earth in FCG.
$\vec{\mathbf{K}}_s$	Source unit vector from the solar system barycenter defined in FCB.
$\vec{\mathbf{b}}$	$= \vec{\mathbf{x}}_2(t_1) - \vec{\mathbf{x}}_1(t_1)$. Baseline vector in FCG.
c	Speed of light
γ	A parameter of linearized PPN metric. $\gamma = 1$ in Einstein's general relativity.
U	$= \sum_p \frac{GM_p}{\lvert\vec{\mathbf{X}}-\vec{\mathbf{X}}_p\rvert c^2}$. Normalized gravitational potential produced by pth body at $\vec{\mathbf{X}}$.
Δt_g	Gravitational delay difference in VLBI observation.

[1] http://gemini.gsfc.nasa.gov/solve/solve.shtml

where $\vec{\mathbf{R}}_{0i}(T_j) = \vec{\mathbf{X}}_0(T_0) - \vec{\mathbf{X}}_i(T_j)$. T_0 is obtained with respect to T_1 by solution of light-time equation

$$\begin{aligned} c(T_1 - T_0) &= |\vec{\mathbf{X}}_1(T_1) - \vec{\mathbf{X}}_0(T_0)| \\ &+(1+\gamma)\sum_J \frac{GM_J}{c^2} \ln \left| \frac{R_{0\mathrm{J}} + R_{1\mathrm{J}}s + R_{01}}{R_{0\mathrm{J}} + R_{1\mathrm{J}} - R_{01}} \right|. \end{aligned} \tag{3}$$

2.2 Definition of source vector for finite distance radio source

In comparison between the two cases, where distance to the radio source is infinite and finite, there is a freedom of choice on source vector for finite distance radio source. Question is which vector in 'finite' case is regarded as correspond to source unity vector in 'infinite' case. Source vector in consensus model ($\vec{\mathbf{K}}_s$) is defined by a unit vector from barycenter of solar system to radio source in FCB. Sovers & Jacobs(1996) used a geocentric vector to radio source to express the difference between delays of plane wave and curved wavefront. Here we suppose unit vector from station 1 to radio source as correspond to source vector in 'infinite' case.

$$\vec{\mathbf{K}}'_s \stackrel{\mathrm{def}}{=} \frac{\vec{\mathbf{R}}_{01}}{R_{01}} \tag{4}$$

The reason of our choice is because delay of VLBI observation is measured at epoch time of signal's arrival to station 1, i.e. station 1 is the reference station. In addition more important reason is that delay characteristic of finite distance radio source departed from that of plane wave is expressed obviously with our definition. That will be seen in latter section.

Figure 1 illustrate a configuration of VLBI observation of curved wavefront. For simplicity, we eliminate any motion of the earth in discussion only in this section. For instance, baseline vector $\vec{\mathbf{B}}$ used here has to be properly denoted $\vec{\mathbf{B}}^*$ (equation (2)) in real condition.

As seen from Figure 1, geometrical difference of distance between R_{02} and R_{01} is given as

$$\begin{aligned} R_{01} - R_{02} = &\ \vec{\mathbf{K}}'_s \cdot \vec{\mathbf{B}} - R_{02}(1 - \cos\theta) \\ = &\ \vec{\mathbf{K}}'_s \cdot \vec{\mathbf{B}} - R_{02}\left(1 - \sqrt{1 - \tfrac{|\vec{\mathbf{K}}'_s \times \vec{\mathbf{B}}|^2}{R_{02}^2}}\right) \\ = &\ \vec{\mathbf{K}}'_s \cdot \vec{\mathbf{B}} - R_{02}\Delta C \end{aligned} \tag{5}$$

The second term of the equation represents the effect of curved wavefront. The factor of this term ΔC can be expressed my Maclaurin expansion as

$$\begin{aligned} \Delta C = &\ \frac{|\vec{\mathbf{K}}'_s \times \vec{\mathbf{B}}|^2}{2R_{02}^2} + \frac{|\vec{\mathbf{K}}'_s \times \vec{\mathbf{B}}|^4}{8R_{02}^4} - \cdots \\ = &\ \sum_{i=1}^{\infty} \frac{|\vec{\mathbf{K}}'_s \times \vec{\mathbf{B}}|^{2i}}{2iR_{02}^{(2i)}}. \end{aligned} \tag{6}$$

3 Difference between the Finite VLBI model and the Consensus model

Time difference measured by VLBI observation in TCB time scale is

$$c(T_2 - T_1) = R_{02}(T_2) - R_{01}(T_1) + c\Delta t_g \tag{7}$$

An approximation $\vec{\mathbf{B}}^* = \vec{\mathbf{B}} + \vec{\mathbf{V}}_2(T_2 - T_1)$ has enough accuracy (better than 6 micro meter on ground based baseline). By using this relation, differential range $R_{02}(T_2) - R_{01}(T_1)$ is expressed with pseudo source vector $\vec{\mathbf{K}}$ defined by equation (2) as:

$$\begin{aligned} R_{02}(T_2) - R_{01}(T_1) &= -\vec{\mathbf{K}}^* \cdot \vec{\mathbf{B}}^* \\ &= -\tfrac{1}{1+\beta_{02}} \vec{\mathbf{K}} \cdot \vec{\mathbf{B}} \end{aligned} \tag{8}$$

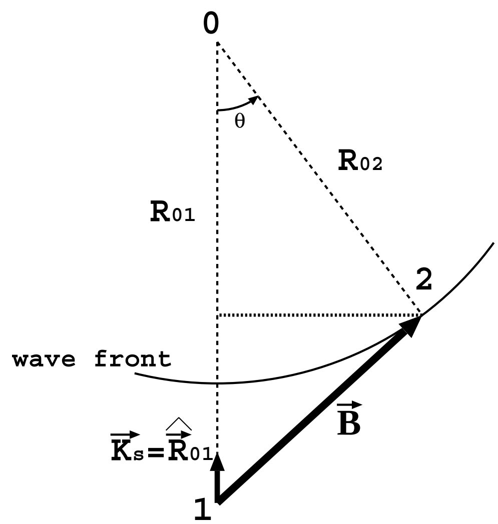

Figure 1: For comparison between the finite VLBI model and the consensus model, $\vec{\mathbf{K}}'_s \stackrel{\text{def}}{=} \widehat{\vec{\mathbf{R}}}_{01}$ is used as definition of source vector corresponding to that in 'infinite' case. Here we are eliminating motion of station 1 and 2 for simplicity. For example, the baseline vector $\vec{\mathbf{B}}$ in this figure must be written as $\vec{\mathbf{B}}^* = \vec{\mathbf{X}}_2(T_2) - \vec{\mathbf{X}}_1(T_1)$ in real condition.

This equation was used to derive the finite distance VLBI model (equation (1)) in other paper [Sekido & Fukushima , 2003]. The same differential range can be expressed with unit source vector $\vec{\mathbf{K}}'_s$ instead of pseudo source vector $\vec{\mathbf{K}}$ as follows:

$$
\begin{aligned}
R_{02}(T_2) - R_{01}(T_1) &= -\vec{\mathbf{K}}'_s \cdot \vec{\mathbf{B}}^* + R_{02}\Delta C \\
&= -\frac{1}{1 + \frac{\vec{\mathbf{K}}'_s \cdot \vec{\mathbf{V}}_2}{c}} \left(-\vec{\mathbf{K}}'_s \cdot \vec{\mathbf{B}} + R_{02}\Delta C \right)
\end{aligned} \tag{9}
$$

A relation between $\vec{\mathbf{B}}$ and $\vec{\mathbf{b}}$ is given (e.g. equation (16) of Sekido & Fukushima (2003)) as:

$$
\vec{\mathbf{B}} = X_2(T_1) - X_1(T_1) = (1 - \gamma U)\vec{\mathbf{b}} - \frac{\vec{\mathbf{V}}_e \cdot \vec{\mathbf{b}}}{c^2} \left(\frac{\vec{\mathbf{V}}_e}{2} + \vec{\mathbf{w}}_2 \right). \tag{10}
$$

And a relation of time interval in TCB ($T_2 - T_1$) and interval of events in TCG is given as (e.g. equation (17) of Sekido & Fukushima (2003)):

$$
T_2 - T_1 = \left(1 + U + \frac{V_e^2}{2c^2} + \frac{\vec{\mathbf{V}}_e \cdot \vec{\mathbf{w}}_2}{c^2} \right) (t_2 - t_1) + \frac{\vec{\mathbf{V}}_e \cdot \vec{\mathbf{b}}}{c^2}. \tag{11}
$$

Substituting equations (9), (10), and (11) into equation (7) becomes

$$
\begin{aligned}
& c\left(1 + U + \frac{V_e^2}{2c^2} + \frac{\vec{\mathbf{V}}_{\mathbf{e}} \cdot \vec{\mathbf{w}}_2}{c^2} \right) (t_2 - t_1) = \\
& \qquad \left(1 + \frac{\vec{\mathbf{K}}'_s \cdot \vec{\mathbf{V}}_2}{c} \right)^{-1} \left[\Delta t_g - (1 - \gamma U)\vec{\mathbf{K}}'_s \cdot \vec{\mathbf{b}} - \frac{\vec{\mathbf{V}}_e \cdot \vec{\mathbf{b}}}{c^2} \left(1 + \frac{\vec{\mathbf{V}}_e \cdot \vec{\mathbf{K}}'_s}{2c} \right) + R_{02}\Delta C \right],
\end{aligned} \tag{12}
$$

where difference $\vec{\mathbf{V}}_2/c$ and $(\vec{\mathbf{V}}_e + \vec{\mathbf{w}}_2)/c$ was eliminated, since it was order of $(V_e/c)^{-3}$.

Time scale of TT (τ) is related with that of TCG (t) by $dt = d\tau(1 - L_G)$, where $L_G = 6.969290134 \times 10^{-10}$ [McCarthy and Petit, 2003]. Consequently, the form of finite VLBI delay in TT scale expressed with unit source vector $\vec{\mathbf{K}}'_s$ is given by

$$c(\tau_2 - \tau_1) = \left(1 + \frac{\vec{\mathbf{K}}'_s \cdot \vec{\mathbf{V}}_2}{c}\right)^{-1}$$
$$\left\{\Delta t_g - \vec{\mathbf{K}}'_s \cdot \vec{\mathbf{b}}[1 - (1+\gamma)U - \frac{V_e^2 + 2\vec{\mathbf{V}}_e \cdot \vec{\mathbf{w}}_2}{2c^2}] - \frac{\vec{\mathbf{V}}_e \cdot \vec{\mathbf{b}}}{c}\left(1 + \frac{\vec{\mathbf{V}}_e \cdot \vec{\mathbf{K}}'_s}{2c}\right) + R_{02}\Delta C'\right\}, \quad (13)$$

where spatial coordinate on the geoid ξ was taken so that speed of light c was kept constant as $d\xi = (1 - L_G)dx$, and baseline vector was re-defined by $\vec{\mathbf{b}} = \vec{\xi}_2(\tau_1) - \vec{\xi}_1(\tau_1)$. And $\Delta C' = (1 - U - L_G - \frac{V_e^2 + 2\vec{\mathbf{V}}_e \cdot \vec{\mathbf{w}}_2}{2c^2})\Delta C$ was used.

The consensus model [Eubanks, 1991; McCarthy and Petit, 2003] is given by

$$c(\tau_2 - \tau_1) = \left[1 + \frac{\vec{\mathbf{K}}_s \cdot (\vec{\mathbf{V}}_e + \vec{\mathbf{w}}_2)}{c}\right]^{-1}$$
$$\left\{c\Delta t_g - \vec{\mathbf{K}}_s \cdot \vec{\mathbf{b}}\left[1 - (1+\gamma)U - \frac{V_e^2 + 2\vec{\mathbf{V}}_e \cdot \vec{\mathbf{w}}_2}{2c^2}\right] - \frac{\vec{\mathbf{V}}_e \cdot \vec{\mathbf{b}}}{c^2}\left(1 + \frac{\vec{\mathbf{K}}_s \cdot \vec{\mathbf{V}}_e}{2c}\right)\right\} \quad (14)$$

Putting the numerator and denominator of right hand side of equation (14) respectively A and B, delay of the consensus model (τ_∞) is $\tau_\infty = A/B$. Regarding $\vec{\mathbf{K}}'_s$ as identical with $\vec{\mathbf{K}}_s$, delay of finite distance VLBI model (τ_F) is written as $\tau_F = \frac{A + R_{02}\Delta C'}{B}$. where difference between $\vec{\mathbf{V}}_2$ and $\vec{\mathbf{V}}_e + \vec{\mathbf{w}}_2$ of the denominators were eliminated, since its factor is $(V_e/c)^{-3}$. Consequently the difference of the two model is given as

$$\tau_F - \tau_\infty = \frac{(1 - U - L_G - \frac{V_e^2 + 2\vec{\mathbf{V}}_e \cdot \vec{\mathbf{w}}_2}{2c^2})}{1 + \frac{\vec{\mathbf{K}}'_s \cdot \vec{\mathbf{V}}_2}{c}} \cdot \frac{R_{02}}{c}\left(1 - \sqrt{1 - \frac{|\vec{\mathbf{K}}'_s \times \vec{\mathbf{B}}^*|^2}{R_{02}^2}}\right) \quad (15)$$

$$= \frac{(1 - U - L_G - \frac{V_e^2 + 2\vec{\mathbf{V}}_e \cdot \vec{\mathbf{w}}_2}{2c^2})}{1 + \frac{\vec{\mathbf{K}}'_s \cdot \vec{\mathbf{V}}_2}{c}} \cdot \frac{R_{02}}{c}\sum_{i=1}^{\infty}\frac{|\vec{\mathbf{K}}'_s \times \vec{\mathbf{B}}^*|^{2i}}{2iR_{02}^{(2i)}}. \quad (16)$$

This difference corresponds to the geometrical delay of curved wavefront multiplied by factor of relativistic time contraction (numerator) and effect of motion of station 2 (denominator). The magnitude of this term is quite large with order of $B^2/2R_{02}$. It reaches up to 50 km in case that the distance to the radio source is 10^9 m and baseline length is 10^7 m, for instance.

A series of VLBI observation campaign were organized in the first half of 2003 for supporting earth swing-by of spacecraft NOZOMI[2]. Several Japanese domestic VLBI stations and Algonquin observatory in Canada joined the campaign. The baseline lengths between Algonquin and Japanese stations were about 9000 km, and NOZOMI approached to the earth in distance 10^9 m or less at the moment of swing-by. Therefore this finite VLBI model was essential for detecting interferometer fringes on the continental baseline in these observations.

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