

# An Order-Sorted Quantified Modal Logic for Ontological Property Classification<sup>1</sup>

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## Abstract

In the field of ontology, several property classifications have been defined as meta-ontologies wherein the properties of individuals are rigorously classified as sortal/non-sortal, rigid/anti-rigid/non-rigid, etc. through philosophical analysis. The notions of such meta-ontologies enable the processing of properties that allow us to reason about taxonomic knowledge in information systems. The goal of our research is to devise a reasoning mechanism that verifies the ontological consistency of knowledge bases in addition to their logical consistency. We first propose an ontological property classification based on individual existence and temporal and situational aspects of properties that establishes an infrastructure for knowledge bases. Second, by combining order-sorted logic and quantified modal logic, we develop an order-sorted modal logic in order to incorporate meta-ontologies into logical reasoning systems that involve rigidity, sortality, and three types of modal operators (temporal/situational/any-world). In the extended logic, sorted expressions and modalities suitably represent the ontological axioms of properties, which are interpreted in different types of possible worlds and in varying domains in Kripke semantics. As a reasoning system of the logic, we present a prefixed tableau calculus to test the satisfiability of sorted modal formulas including the ontological property classification.

## 1 Introduction

Formal ontology deals with types of entities in the real world, such as properties, events, processes, objects, and parts, as discussed in Smith's paper [25]. In this field, Guarino and Welty [15] have defined meaningful property classifications as meta-ontologies wherein the properties of individuals are rigorously classified as sortal/non-sortal, rigid/anti-rigid/non-rigid, etc., through philosophical analysis. These meta-ontological classifications describe the general features of knowledge, which can enable the processing of properties that allows us to reason about taxonomic knowledge in information systems.

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<sup>1</sup>This paper is an extended version of [20].

On the other hand, order-sorted logic has been recognized as a useful tool for providing logical reasoning systems on taxonomic knowledge [5, 29, 23, 8, 12, 27, 30, 18]. Kaneiwa and Mizoguchi [19] observed that the ontological property classification [31] fits the formalization of order-sorted logic, and they refined the sorted logic by means of the ontological notions of rigidity and sortality. *Rigid* properties are defined as stable (i.e., if an individual has a rigid property, it is true in any possible world), and *sortal* properties are defined as distinguishable. However, these approaches did not cover *individual existence* and *temporal and situational aspects of properties* for taxonomic knowledge (only certain temporal aspects were axiomatized by the modal and tense operators [14]).

The first aim of this paper is to present an ontological property classification extended to include the notions of individual existence and time, situation, and time-situation dependencies, which are based on the following.

- Ontologically, physical entities have a property as long as they exist although they will cease to exist at some time.
- One property (e.g., baby) holds depending only on time and is situationally stable, while another (e.g., weapon) holds depending on its use in a situation and is temporally stable. For example, a knife can be a weapon in one situation but is usually employed as a tool for eating.

In particular, we consider that these notions are derived from the ontological classification of properties extended by the following two ways:

- (i) by defining the rigidity of properties together with individual existence and
- (ii) by further classifying anti-rigid properties based on time and situation dependencies.

To establish them in logic, we make use of the semantics of quantified modal and temporal logics. The modeling of the extensions should include varying domains and distinguish times and situations from other possible worlds in Kripke semantics. Although the modal and temporal logics are usually formalized in *constant domains*, several quantified modal logics address philosophical problems such as *varying domains*, *non-rigid terms*, and local terms. Garson [11] discussed the different systems for variants of quantified modal logics. Fitting and Mendelsohn [7] treated the rigidity of terms and constant/varying domains by means of a tableau calculus and a predicate abstraction. Cialdea-Mayer and Cerrito [4] proposed a prefixed tableau calculus for all the variants of quantified modal logics with respect to cumulative/varying domains, rigid/non-rigid terms, and local/non-local terms.

Our second aim is to present an order-sorted modal logic that includes the refined property classification. This logic requires a combination of order-sorted logic, quantified modal logic, and temporal logic due to their respective features.

1. Order-sorted logic has the advantage that sorted terms and formulas adequately represent properties based on the ontological property classification.
2. Cialdea-Mayer and Cerrito's quantified modal logic provides us with a prefixed tableau calculus for supporting varying domains and non-rigid terms, which can be extended to order-sorted terms/formulas and multi-modal operators.

3. Temporal logic contains time representation; however, the standard reasoning systems are propositional [24, 13], or the first-order versions [10, 9, 3] have constant domains since they are not easily extended to the first-order temporal logic with varying domains [11].

The proposed logic includes a logical reasoning system for verifying the ontological and logical consistency of knowledge bases with respect to properties. Unary predicates and sorts categorized on the basis of rigidity, sortality, and time and situation dependencies represent properties in sorted modal formulas, appropriately interpreted with different modalities and varying domains. We extend a prefixed tableau calculus (proposed by Cialdea-Mayer and Cerrito) that tests the satisfiability of such sorted modal formulas comprising individual existence and three types of modal operators (temporal/situational/any-world). In particular, we denote the types of possible worlds in prefixed formulas that are derived by rule application in the calculus. Moreover, we create new inference rules (modal operator, sorted quantifier, sort predicate, and existential predicate rules) that handle sorted expressions and multi-modalities supporting individual existence.

This paper is arranged as follows. In Section 2, we formally define the notions of a property and its rigidity, sortality, and time and situation dependencies, which are used to refine an ontological property classification in Kripke semantics. Regarding the ontological property classification, Section 3 formalizes the syntax and semantics of an order-sorted quantified modal logic. In Section 4, we extend a prefixed tableau calculus to our proposed sorted modal logic, which enables us to check the consistency of property assertions in logical formulas. Section 5 discusses the completeness of the tableau calculus. Finally, in Section 6, we offer our conclusion and discuss future work.

## 2 Property Classification in Semantics

### 2.1 Property and Sortality

The term *property* should be cautiously used since it has multiple meanings. In logic, a unary predicate denotes a *property*. That is, *human* in the predicate formula *human*(*x*) is a property just like *red* in the predicate formula *red*(*x*). In the Web Ontology Language OWL [26, 6, 22], links between two nodes are referred to as a property, since OWL is designed by incorporating the concepts used in a semantic network into its conceptual framework. It could be stated that the conceptualization of OWL might lead users to a lower level of abstraction at which nodes and links are the main conceptual constructs that are close to an implementation level. A property in OWL corresponds to a binary predicate, whereas it corresponds to a unary predicate in logic. In the area of artificial intelligence, there is a clear distinction between *property* and *object*. The former is necessarily associated with the latter which in turn cannot exist without any property. For example, a *human* is an object and *tall* is a property. The use of property in the field of ontological engineering is fairly similar to that used in AI. Furthermore, many believe that ontology has to be representation-independent, even of logical expression. Logicians might consider *x* in *p*(*x*) to represent an individual before class identification; thus, there is no difference in kind between *human* and *red* because both are a predication of what *x* is. Therefore, we use the term *property* as a unary predicate and reconsider the meaning of properties on the basis of ontological analysis.

Strawson discussed the *sortality* of properties [28]. As cited by Guarino, Carrara, and Giaretta [14], a sortal property “supplies a principle for distinguishing and counting individual particulars which it collects.” Hence, to embed *the principle for distinguishing individual particulars* in the definition of a sortal property, we need to introduce the notion of identity conditions (ICs) [28, 16]. Let a sameness formula be given as  $\Sigma_\chi(x, y, t, t') =_{def} \forall z(\chi(x, z, t) \leftrightarrow \chi(y, z, t'))$  where  $\chi$  is a suitable characteristic relation for a property  $p$ . As defined by Guarino and Welty, a property  $p$  carries an IC if the following holds.

$$\begin{aligned} E(x, t) \wedge E(y, t') \wedge p(x, t) \wedge p(y, t') &\rightarrow (\Sigma_\chi(x, y, t, t') \leftrightarrow x = y) \\ &\neg \forall xy(\Sigma_\chi(x, y, t, t') \leftrightarrow x = y) \end{aligned}$$

The first formula implies that if two individuals  $x$  and  $y$  exist at times  $t$  and  $t'$ , which are both instances of the property  $p$ , then they have the same characteristics if and only if  $x = y$ . The second formula is a non-triviality constraint that excludes trivial cases. For example, suppose that  $\chi$  denotes the ‘has-parts’ relationship, the property *human* carries an IC since for any two humans, if they consist of the same parts, they are identical. Such ICs are well discussed [16, 2] as one of the main topics on formal ontology<sup>1</sup>.

A property is *sortal* if it carries an IC (i.e., distinguishable) and any proper part of each non-atomic entity does not have the same property (i.e., countable without any container). For example, the property *person* carries an IC and proper parts of a *person* are not a person. In a logical language, a sortal property is known as a *sort* if it is used to denote the domain or range of a variable, function, or predicate. In Kripke semantics, a set  $W$  of possible worlds  $w_i$  is introduced and properties are interpreted differently in each world. Let  $U$  be the set of individuals (i.e., the universe), and let  $I = \{I_w \mid w \in W\}$  be the set of interpretation functions  $I_w$  for all possible worlds  $w \in W$ . We can now specify that every sort  $s$  is interpreted by  $I_w(s) \subseteq U$  for each world  $w$ , and in building a sort-hierarchy, a subsort relation  $s_i < s_j$  (e.g., *boy* < *human*) is interpreted by  $I_w(s_i) \subseteq I_w(s_j)$ . A unary predicate  $p_s$  denoted by a sort  $s$  is called a sort predicate. The interpretation of a sort  $s$  determines the interpretation of the corresponding sort predicate  $p_s$ . The sort predicate represents the statement that a term  $t$  belongs to a sort by the predicate formula  $p_s(t)$  or it does not belong to the sort by the negative predicate formula  $\neg p_s(t)$ . Standard order-sorted logic does not include the intensional semantics that reflects the rigidity of sorts and sort predicates. Therefore, ideally, every sort  $s$  and its sort predicate  $p_s$  are identical in their interpretation and are defined by  $I_w(s) = I_w(p_s)$  in each world  $w \in W$ .

A sort is a *substantial sort* (called a *type*) if it is rigid. For example, the sort *human* is a type (i.e., rigid) since every human is always a human, but the sort *student* is anti-rigid since each student will not have the property after the student graduates from school.

## 2.2 Rigidity

Let us define the *rigidity* of properties in Kripke semantics. For the sake of simplicity, we can assume the constant domains (i.e., the domains that are the same in all possible worlds) and the unique name assumption (i.e., if two individuals have different names, they are different

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<sup>1</sup>Kaplan [21] shows that Guarino and Welty’s definition cannot completely exclude trivial cases. Hence, this paper does not treat the properties that give rise to the problem because our aim is not to completely define identity conditions.

individuals). Let  $\tau$  be a type (as a substantial sort). The rigidity of types is defined by the property that for all possible worlds  $w_i, w_j \in W$ ,  $I_{w_i}(\tau) = I_{w_j}(\tau)$ . This leads to the rigidity of individuals denoted by constants and functions. Let  $c$  be a constant and  $f$  be a function. Then, for all possible worlds  $w_i, w_j \in W$ ,  $I_{w_i}(c) = I_{w_j}(c)$  and  $I_{w_i}(f) = I_{w_j}(f)$  hold. However, the constant domains do not appear to be realistic since there may be different entities (individuals) in each possible world. For example, every instance of the property *person* ceases to exist at some time since no person can live forever. To capture the semantics, we select varying domains such that  $U_w$  is the set of individuals existing in a possible world  $w$ . The domains enable us to consider the case where  $U_{w_1}$  and  $U_{w_2}$  do not coincide for some possible worlds  $w_1, w_2 \in W$ .

The rigidity of sorts, constants, and functions is redefined by supporting individual existence in the following manner.

**Definition 1 (Existential Rigidity)** *Let  $c$  be a constant and  $f$  be an  $n$ -ary function. For all possible worlds  $w_i, w_j \in W$  and any interpretation functions  $I_{w_i}$  and  $I_{w_j}$  for  $w_i$  and  $w_j$ ,*

1. *if  $I_{w_i}(c), I_{w_j}(c) \in U_{w_i} \cap U_{w_j}$ , then  $I_{w_i}(c) = I_{w_j}(c)$ , and*
2. *for  $d_1, \dots, d_n \in U_{w_i} \cap U_{w_j}$ , if  $\{I_{w_i}(f)(d_1, \dots, d_n), I_{w_j}(f)(d_1, \dots, d_n)\} \subseteq U_{w_i} \cap U_{w_j}$ , then  $I_{w_i}(f)(d_1, \dots, d_n) = I_{w_j}(f)(d_1, \dots, d_n)$ .*

*Let  $w \in W$ , let  $I_w$  be an interpretation function for  $w$ , let  $d \in U_w$ , and let  $R'$  be an accessibility relation over  $W \times W$ . Then, the following holds:*

3. *for every type  $\tau$ , if  $d \in I_w(\tau)$  and  $(w, w') \in R'$ , then  $d \in U_{w'}$  implies  $d \in I_{w'}(\tau)$ , and*
4. *for every anti-rigid sort  $\sigma$ , if  $d \in I_w(\sigma)$ , then there exists  $w_j \in W$  with  $(w, w_j) \in R'$  such that  $d \notin I_{w_j}(\sigma)$  with  $d \in U_{w_j}$ .*

### 2.3 Time and Situation Dependencies

The semantics can be further elaborated in terms of the dependencies of time and situation. The *time* and *situation* are treated as special possible worlds when defining distinctions among anti-rigid sorts (as non-substantial properties). We now introduce the set  $W_{\text{Tim}}$  of times  $tm_i$  and the set  $W_{\text{Sit}}$  of situations  $st_i$  where  $W_{\text{Tim}} \cup W_{\text{Sit}} \subseteq W$ . Because  $W_{\text{Tim}}$  and  $W_{\text{Sit}}$  are subsets of  $W$ , they do not violate rigidity in their interpretation if the types, constants, and functions preserve their rigidity in any time and situation. We show dependencies on time and situation that classify anti-rigid sorts as follows:

- time dependent:** baby, child, youth, adult, elderly
- situation dependent:** weapon, table, student
- time-situation dependent:** novice teacher

In Figure 1, these distinctions are added to the property classification. The time dependency implies that the meaning of a property depends only on time or is decided essentially by time. For example, the property *baby* is time dependent, so that each instance has the denoting property in a particular time or period.

The situation dependency indicates that a situation establishes whether a property holds but time does not. Moreover, the situation dependency obtained from extending types

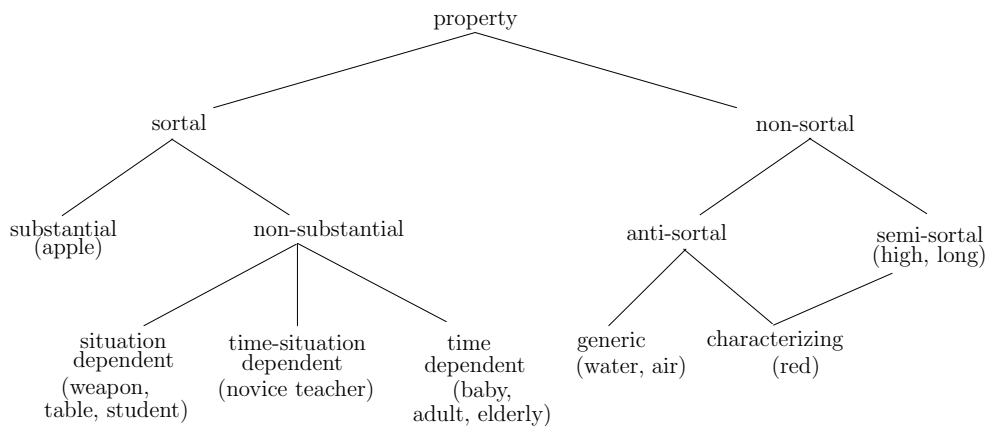


Figure 1: Ontological property classification

(such as *weapon*, *table*, but not *student*) involves a complex idea, as described below. We can regard the property *weapon* as a substantial sort (type); however, it is anti-rigid and situation-dependent if it is used as a role expressed by the sort predicate  $p_{\text{weapon}}$ . For instance, the properties *weapon* and *table* have two types of instances:

- (i) guns and dining tables that innately possess the property of *weapon* and *table* and
- (ii) knives and boxes that play the roles of *weapon* and *table*, respectively.

In the latter case, the properties are not really the aforementioned artifacts and are just referred to as *weapon* and *table*. Thus, knives play the role of a *weapon* only when they are used to attack or kill someone. In the language of order-sorted logic, the former case is an instantiation of a sort (e.g.,  $c_{\text{weapon}}$ ), and the latter case is an entity characterized by a sort predicate (e.g.,  $p_{\text{weapon}}(c)$ ). In other words, the type *weapon* is rigid but the property  $p_{\text{weapon}}$  is anti-rigid (situation dependent). It guarantees that all types are still rigid even if an entity additionally belongs to the predicate denoted by a type (e.g., a knife is being used as a weapon but not essentially a weapon). These different interpretations for a property are suitable for us because we also interpret a natural language word (e.g., *weapon*) as having an essential property and a role. This consideration gives the following definition which disproves the fact that sorts and their sort predicates are interpreted identically in the semantics.

**Definition 2** *Let  $\tau$  be a type. For every  $w \in W$  and any interpretation function  $I_w$  for  $w$ , if the type predicate  $p_\tau$  is situation dependent, then  $I_w(\tau) \subsetneq I_w(p_\tau)$ .*

For example,  $gun1 \in I_w(\text{weapon}) \cap I_w(p_{\text{weapon}})$ , whereas  $knife1 \notin I_w(\text{weapon})$  and  $knife1 \in I_w(p_{\text{weapon}})$ .

Here, we define those dependencies semantically in possible worlds. The basic notion of interpreting time dependency is that for every time-dependent property  $p$  and for every individual  $d \in U$ , if  $d \in I_{tm}(p)$  with  $tm \in W_{\text{Tim}}$ , then another time  $tm_j \in W_{\text{Tim}}$  exists such that  $d \notin I_{tm_j}(p)$ . This simple definition is based on the constant domains, which can be refined by considering the existence of individuals in each world. In the following, the varying domains will specify time, situation, and time-situation dependencies where

$R_{\text{Tim}} \subseteq W \times W_{\text{Tim}}$  and  $R_{\text{Sit}} \subseteq W \times W_{\text{Sit}}$  are accessibility relations from worlds to times and situations, respectively.

**Definition 3 (Time Dependency)** *A unary predicate  $p$  is time-dependent if the following statements hold:*

1. (*temporally unstable*) for all  $(w, tm) \in R_{\text{Tim}}$  and for all  $d \in U_{tm}$ , if  $d \in I_{tm}(p)$ , then there exists  $tm_j \in W_{\text{Tim}}$  with  $(tm, tm_j) \in R_{\text{Tim}}$  such that  $d \notin I_{tm_j}(p)$  with  $d \in U_{tm_j}$ .
2. (*situationally stable over time*) for all  $w \in W$  and for all  $tm \in W_{\text{Tim}}$ , if  $d \in I_{tm}(p)$  with  $(w, tm) \in R_{\text{Tim}}$ , then for all situations  $st \in W_{\text{Sit}}$  with  $(tm, st) \in R_{\text{Sit}}$ ,  $d \in U_{st}$  implies  $d \in I_{st}(p)$ .

The *temporally unstable* implies that for every time  $tm$  accessible from a world  $w$ , if an individual  $d$  has the property  $p$  at the time  $tm$ , we can find a time  $tm_j$  accessible from  $tm$  where the individual does not have the property. The *situationally stable under time* defines the fact that for every time  $tm$  accessible from a world  $w$ , if an individual  $d$  has the property  $p$  at the time  $tm$ , then the individual has this property in any situation  $st$  accessible from the time  $tm$  as long as the individual exists. Similar to the above time dependency, situation dependency can be defined as follows:

**Definition 4 (Situation Dependency)** *A unary predicate  $p$  is situation-dependent if the following statements hold:*

1. (*situationally unstable*) for all  $(w, st) \in R_{\text{Sit}}$  and for all  $d \in U_{st}$ , if  $d \in I_{st}(p)$ , then there exists  $st_j \in W_{\text{Sit}}$  with  $(st, st_j) \in R_{\text{Sit}}$  such that  $d \notin I_{st_j}(p)$  with  $d \in U_{st_j}$ .
2. (*temporally stable under situation*) for all  $w \in W$  and for all  $st \in W_{\text{Sit}}$ , if  $d \in I_{st}(p)$  with  $(w, st) \in R_{\text{Sit}}$ , then for all times  $tm \in W_{\text{Tim}}$  with  $(st, tm) \in R_{\text{Tim}}$ ,  $d \in U_{tm}$  implies  $d \in I_{tm}(p)$ .

Moreover, the time-situation dependency is defined such that the meaning of a property depends on both the time and situation. For example, the property *novice\_teacher* is time-situation dependent. Since each novice teacher will become a veteran teacher after a number of years, the property holds only at a particular time under a situation. In other words, the time-situation dependency implies the time dependency under a situation, while the situation dependency bears no relationship to time. In the semantics with varying domains, we define the time-situation dependency as a more complex dependency, as shown below.

**Definition 5 (Time-Situation Dependency)** *A unary predicate  $p$  is time-situation dependent if the following statements hold:*

1. (*situationally unstable*) the same as in Definition 4.
2. (*temporally unstable under situation*) for all  $w \in W$  and for all  $st \in W_{\text{Sit}}$ , if  $d \in I_{st}(p)$  with  $(w, st) \in R_{\text{Sit}}$ , then there are some  $tm_i, tm_j \in W_{\text{Tim}}$  with  $(st, tm_i), (st, tm_j) \in R_{\text{Tim}}$  such that  $d \in I_{tm_i}(p)$  and  $d \notin I_{tm_j}(p)$  with  $d \in U_{tm_i} \cap U_{tm_j}$ .

Besides the situational unstability, the *temporally unstable under situation* implies that for every situation  $st$  accessible from a world  $w$ , if an individual  $d$  has the property  $p$  in the situation  $st$ , then there are times  $tm_i, tm_j$  accessible from  $st$  such that the individual has the property  $p$  at  $tm_i$  but not at  $tm_j$ . The time-situation dependency is motivated by the fact that either one or the other of the time and situation dependencies cannot explain time-situation dependent properties (such as the novice teacher). This is because the time-situation dependency is different from both the time and situation dependencies holding.

The interpretation of such dependencies in possible worlds semantically determines partial rigidity as the ontological differences among anti-rigid properties. That is, even if a property is anti-rigid, it may be rigid as a particular type of possible worlds (e.g., times or situations), as a subset of  $W$ . For example, the property *baby* is time dependent, i.e., it is temporally unstable and situationally stable over time. When Bob is a baby, the time dependency derives the fact that Bob is a baby in any situation within the time. Formally, if  $bob \in I_{tm}(baby)$ , then for any situation  $st$  accessible from  $tm$ ,  $bob \in U_{st}$  implies  $bob \in I_{st}(baby)$  (i.e., if Bob exists in  $st$ , he is a baby). The property can be viewed as rigid over situations, while it is anti-rigid (or partially rigid) in any world. Notice that the formal ontology deals with the essential nature of properties by ignoring some exceptions. Thus, the time instability of *baby* typically holds even if a baby could die while still a baby. Similarly, in taxonomic knowledge, birds can typically fly even if penguins cannot fly.

## 2.4 Non-Sortality

We further deal with the *non-sortality* of properties that decides whether a property is sortal or not in the knowledge representation of knowledge bases. As previously mentioned, a property is sortal if it carries an IC and any proper part of each non-atomic entity does not have the same property; it is non-sortal otherwise. Furthermore, non-sortal properties are classified into anti-sortal and semi-sortal. A non-sortal property is anti-sortal if all the parts have the same property as a whole; it is semi-sortal if a part has the same property as a whole. The differences between sortal and non-sortal properties (anti-sortal and semi-sortal properties) lead to the following identifiability. Any instance of each sortal property is essentially identifiable (e.g., we can question how many instances of *person* exist); however, an anti-sortal property is not identifiable but countable by *using a container* (e.g., there are five cups of *water* as a mass or unit) since all the parts of each instance have the same property. That is, part of the water is also water so an instance of the water and its part cannot be identified or counted in the absence of a container. In the philosophical tradition, mass properties like water are generally counted as sortals, although they are not countable, because they do have an identity criterion. To consistently follow the tradition in our sortality, we distinguish between the two properties *water* and *cup\_of\_water*. Then, the property *water* is not sortal, but the property *cup\_of\_water* is sortal because the proper part of a cup of water is not a cup of water.

**sortal (identifiable):** person, student, cup of water

**anti-sortal (countable with a container):** water, air

Consider the two non-sortal properties *water* and *large*. The former is anti-sortal, since any part of *water* is also water. The property *large* is semi-sortal, since it has a part that is not large.

**semi-sortal (difficult to count):** large, long

This type of property is regarded as being difficult to count because there is an ambiguous boundary to distinguish between the properties as a whole. For example, a half of large things may be large, but a twentieth part of them may not be large.

Figure 1 shows the classification of anti-sortal properties into rigid and non-rigid. An anti-sortal property is called generic if it is rigid. Moreover, a non-sortal property (an anti-sortal or semi-sortal property) is said to be characterizing if it is non-rigid. The following examples are generic, characterizing, and semi-sortal properties.

**generic (rigid and anti-sortal):** water, air  
**characterizing (non-rigid and non-sortal):** red  
**semi-sortal (rigid or non-rigid):** large, long

The property *air* is generic as a rigid and anti-sortal property, since it holds in any possible world. The property *red* is characterizing, since some red thing can change the color and there is a red thing whose part is red. For example, a red car can be repainted and part of blood is also red. Furthermore, the features of anti-sortal properties lead to a semantic constraint with respect to a part-of relation ( $\prec$ ) as follows.

**Definition 6 (Part-of Relation for Anti-Sortal)** *Let  $p$  be an anti-sortal predicate, let  $w \in W$ , let  $I_w$  be an interpretation function for  $w$ , and let  $d_1, d_2 \in U_w$ . If  $d_1 \in I_w(p)$  and  $(d_2, d_1) \in I_w(\prec)$ , then  $d_2 \in I_w(p)$ .*

For example, if  $d \in I_w(p)$  (a cup of liquid  $d$  is *water*), then  $d' \prec d$  implies that  $d' \in I_w(p)$  (part  $d'$  of  $d$  is also *water*).

### 3 Order-Sorted Modal Logic

The purpose of this paper is to incorporate the ontological property classification into the syntax and semantics of an order-sorted modal logic. In the logic, various properties are expressed by unary predicates and sorts where predicates are semantically related to sorts if their arities are unary. The rigidity of properties (and individuals) interpreted in Kripke semantics is axiomatized by formulas with modal operators due to the characterization of temporal and situational dependences. Therefore, we need to introduce the three types of modal operators  $\blacksquare, \blacklozenge$  (any world),  $\square_{\mathbf{Tim}}, \diamond_{\mathbf{Tim}}$  (temporal), and  $\square_{\mathbf{Sit}}, \diamond_{\mathbf{Sit}}$  (situational) in the logical language. Specifically, these operators are used to express property assertions as modal formulas, whose logical and ontological consistencies are verified in an extended tableau calculus.

We define the syntax and semantics of an order-sorted modal logic as follows.

#### 3.1 Syntax

The alphabet of a sorted first-order modal language  $\mathcal{L}$  with rigidity and sort predicates comprises the following symbols: a countable set  $T$  of type symbols (including the greatest type  $\top$ ), a countable set  $S_A$  of anti-rigid sort symbols ( $T \cap S_A = \emptyset$ ), a countable set  $C$  of constant symbols, a countable set  $F_n$  of  $n$ -ary function symbols, a countable set  $P_n$  of

$n$ -ary predicate symbols (including the existential predicate symbol  $E$ ; the set  $P_{T \cup S_A}$  of sort predicate symbols  $\{p_s \mid s \in T \cup S_A\}$ ; and a countable set  $P_{non}$  of non-sortal predicate symbols), the connectives  $\wedge, \vee, \rightarrow, \neg$ , the modal operators  $\Box_i, \Diamond_i, \blacksquare, \blacklozenge$ , and the auxiliary symbols  $(, )$ .

We generally refer to type symbols  $\tau$  or anti-rigid sort symbols  $\sigma$  as *sort symbols*  $s$ .  $T \cup S_A$  is the set of sort symbols.  $V_s$  denotes an infinite set of variables  $x_s$  of sort  $s$ . We abbreviate variables  $x_\top$  of sort  $\top$  as  $x$ . The set of variables of all sorts is denoted by  $V = \bigcup_{s \in T \cup S_A} V_s$ . The unary predicates  $p_s \in P_1$  indexed by the sorts  $s$  (called *sort predicates*) are introduced for all sorts  $s \in T \cup S_A$ . In particular, the predicate  $p_\tau$  indexed by a type  $\tau$  is called a *type predicate*, and the predicate  $p_\sigma$  indexed by an anti-rigid sort  $\sigma$  is called an *anti-rigid sort predicate*. Hereafter, we assume that the language  $\mathcal{L}$  contains all the sort predicates in  $P_{T \cup S_A}$ . Types can be situation dependent (i.e., no type has time dependency or time-situation dependency), while anti-rigid sorts can be either time, situation, or time-situation dependent. For example, the type *weapon* is situation dependent, and the anti-rigid sort *adult* is time dependent. Each sort predicate  $p_s$  has the same dependency as its sort  $s$ .

**Definition 7 (Sorted Signatures)** *A signature of a sorted first-order modal language  $\mathcal{L}$  with rigidity and sort predicates (called sorted signature) is a tuple  $\Sigma = (T, S_A, \Omega, \leq)$  such that*

1.  $(T \cup S_A, \leq)$  is a partially ordered set of sorts where  $T \cup S_A$  is the union of the set of type symbols and the set of anti-rigid sort symbols in  $\mathcal{L}$  and each ordered pair  $s_i \leq s_j$  is a subsort relation (implying that  $s_i$  is a subsort of  $s_j$ ) fulfilling the following:
  - (a) every subsort of types is a type or an anti-rigid sort ( $s \leq \tau$ ) and every subsort of anti-rigid sorts is an anti-rigid sort ( $\sigma \leq \sigma'$ );
  - (b) every subsort of time dependent sorts is time or time-situation dependent;
  - (c) every subsort of situation dependent sorts is situation or time-situation dependent; and
  - (d) every subsort of time-situation dependent sorts is time-situation dependent,
2.  $\Omega$  is a set of sort declarations of constants, functions, and predicates where
  - (a) if  $c \in C$ , then there is a unique constant declaration  $c: \rightarrow \tau \in \Omega$ ;
  - (b) if  $f \in F_n$  ( $n > 0$ ), then there is a unique function declaration  $f: \tau_1 \times \cdots \times \tau_n \rightarrow \tau \in \Omega$ ; and
  - (c) if  $p \in P_n$ , then there is a unique predicate declaration  $p: s_1 \times \cdots \times s_n \in \Omega$  (in particular, if  $p_s \in P_{T \cup S_A}$ , then there is a unique sort predicate declaration  $p_s: \tau \in \Omega$  where  $s \leq \tau$ , and if  $p \in P_{non}$ , then  $p: undef \in \Omega$ ).

A partially ordered set  $(T \cup S_A, \leq)$  constructs a sort-hierarchy by suitably ordering different types of sorts where the subsort relation is assumed to be transitive and reflexive. That is, the sort-hierarchy is restricted by Definition 7 as in the following:

- no subsort of anti-rigid sorts is a type,
- no subsort of situation dependent sorts nor time-situation dependent sorts is time dependent, and

- no subsort of time dependent sorts nor time-situation dependent sorts is situation dependent.

These restrictions are ontologically guaranteed by the fact that each sort inherits (temporal and situational) unstability and anti-rigidity from its supersorts. For example, the sort *novice\_teacher* must be situationally unstable (as time-situation dependent) if the supersort *teacher* is situation dependent.

In sorted signatures, the sorts of constants, functions, and predicates have to be declared by adhering to the rigidity in Definition 1, i.e., since every constant and function is rigid, their sorts have to be rigid. The sort declarations of constants  $c$  and functions  $f$  are denoted by the forms  $c: \rightarrow \tau$  and  $f: \tau_1 \times \dots \times \tau_n \rightarrow \tau$  where types  $\tau_i, \tau$  are used to declare the sorts. Constants and functions are required to be rigidly sorted in order to avoid the anti-rigid domains and ranges of constants and functions. On the other hand, the sort declarations of predicates are denoted by the form  $p: s_1 \times \dots \times s_n$  where types and anti-rigid sorts  $s_i$  can be used to set the domains of the predicates  $p$ . This is because the domains of predicates can be anti-rigid.

Although the declarations of sort predicates are defined by the greatest sort  $\top$  (i.e.,  $p_s: \top$ ) in [1], we reconsider it in this paper. For each anti-rigid sort  $\sigma$ , there is a basic type  $\tau$  to be an entity of  $\sigma$ , i.e., every entity of the sort  $\sigma$  must be an entity of the type  $\tau$ . For example, the anti-rigid sorts *student* and *husband* respectively have the basic types *person* and *male\_person*, defined as the necessary condition of being able to play the roles. Hence, the declaration of each sort predicate  $p_s$  is defined by a type  $\tau$  such that  $s \leq \tau$  (i.e.,  $p_s: \tau$ ) if it is anti-rigid. Unlike anti-rigid predicates, the declaration of a type predicate  $p_{\tau'}$  is simply defined by a necessary condition for the predicate, that is a supersort of the target type (i.e.,  $p_{\tau'}: \tau$  declares the domain of  $p_{\tau'}$  where  $\tau' \leq \tau$ ). The supersort is useful in knowledge representation because we may state whether or not a sorted term with  $\tau$  belongs to a type predicate  $p_{\tau'}$ . For example, the type *person* may have a necessary condition *animal*. In order to state that the sorted constant  $c_{animal}$  belongs to (or does not belong to) the type *person* (e.g.,  $p_{person}(c_{animal})$  or  $\neg p_{person}(c_{animal})$ ), the type predicate  $p_{person}$  should be declared by the form  $p_{person}: animal$ .

On the contrary, there is no basic type if a unary predicate corresponds to a non-sortal property. For instance, a necessary condition of the property *red* appears to be *physical\_thing*. However, when considering *red* as the necessary condition of *red\_light*, *light* is a thing of some kind, and it is debatable whether light has color in the same way that material objects have color. Moreover, the property *water* may have the general property *substance*, but it is a non-sortal property (not a type). In order to avoid these problems, we express every non-sortal property by a unary predicate (in  $P_{non}$ ) without a particular sort declaration (denoted instead by  $p: undef$ ). Further, the part-of relation  $\prec$  is a binary predicate declared by  $\prec: undef \times undef \in \Omega$  since it is used to describe a relation among the entities embodying a non-sortal property.

Following the sorted signature, we introduce the three types of terms: *typed term*, *anti-rigid sorted term*, and *sorted term* in a sorted first-order modal language  $\mathcal{L}_\Sigma$ .

**Definition 8 (Typed Terms)** *Let  $\Sigma = (T, S_A, \Omega, \leq)$  be a sorted signature. The set  $\mathcal{T}_\tau^-$  of terms of type  $\tau$  (called typed terms) is the smallest set such that*

1. for every  $x_\tau \in V_\tau$ ,  $x_\tau \in \mathcal{T}_\tau^-$ ,

2. for every  $c \in C$  with  $c: \rightarrow \tau \in \Omega$ ,  $c_\tau \in \mathcal{T}_\tau^-$ ,
3. if  $t_1 \in \mathcal{T}_{\tau_1}^-, \dots, t_n \in \mathcal{T}_{\tau_n}^-, f \in F_n$ , and  $f: \tau_1 \times \dots \times \tau_n \rightarrow \tau \in \Omega$ , then  $f_{\tau^*, \tau}(t_1, \dots, t_n) \in \mathcal{T}_\tau^-$  with  $\tau^* = \tau_1, \dots, \tau_n$ , and
4. if  $t \in \mathcal{T}_{\tau'}^-$  and  $\tau' \leq \tau$ , then  $t \in \mathcal{T}_\tau^-$ .

**Definition 9 (Anti-Rigid Sorted Terms)** Let  $\Sigma = (T, S_A, \Omega, \leq)$  be a sorted signature. The set  $\mathcal{T}_\sigma^-$  of terms of anti-rigid sort  $\sigma$  (called anti-rigid sorted terms) is the smallest set such that

1. for every  $x_\sigma \in V_\sigma$ ,  $x_\sigma \in \mathcal{T}_\sigma^-$  and
2. if  $t \in \mathcal{T}_{\sigma'}^-$  and  $\sigma' \leq \sigma$ , then  $t \in \mathcal{T}_\sigma^-$ .

**Definition 10 (Sorted Terms)** Let  $\Sigma = (T, S_A, \Omega, \leq)$  be a sorted signature. The set  $\mathcal{T}_s$  of terms of sort  $s$  (called sorted terms) is the smallest set such that

1.  $\mathcal{T}_s^- \subseteq \mathcal{T}_s$  and
2. if  $t \in \mathcal{T}_{s'}$  and  $s' \leq s$ , then  $t \in \mathcal{T}_s$ .

Due to the rigidity of types and anti-rigid sorts, any anti-rigid sorted term (in  $\mathcal{T}_\sigma^-$ ) must be a variable term whereas typed terms (in  $\mathcal{T}_\tau^-$ ) can contain constants and functions. In other words, every anti-rigid sorted term is not rigid (e.g.,  $x_{student}$ ) and every typed term is rigid (e.g.,  $c_{person}$ ). We denote  $sort(t)$  as the sort of a term  $t$ , i.e.,  $sort(t) = s$  if  $t$  is of the form  $x_s$ ,  $c_s$ , or  $f_{\tau^*, s}(t_1, \dots, t_n)$ . Next, we define the set of sorted modal formulas in the language  $\mathcal{L}_\Sigma$ .

**Definition 11 (Sorted Formulas)** Let  $\Sigma = (T, S_A, \Omega, \leq)$  be a sorted signature. The set  $\mathcal{F}$  of formulas is the smallest set such that

1. if  $t_1 \in \mathcal{T}_{s_1}, \dots, t_n \in \mathcal{T}_{s_n}$ ,  $p \in P_n$ , and  $p: s_1 \times \dots \times s_n \in \Omega$ , then  $p(t_1, \dots, t_n)$  is a formula,
2. if  $t \in \mathcal{T}_\tau$ ,  $p \in P_{T \cup S_A}$ , and  $p_s: \tau \in \Omega$ , then  $p_s(t)$  is a formula,
3. if  $t \in \mathcal{T}_\top$ , then  $E(t)$  and  $p(t)$  are formulas where  $p \in P_{non}$ , and
4. if  $F, F_1$ , and  $F_2$  are formulas, then  $\neg F$ ,  $(\forall x_s)F$ ,  $(\exists x_s)F$ ,  $\Box_i F$ ,  $\Diamond_i F$ ,  $\blacksquare F$ ,  $\blacklozenge F$ ,  $F_1 \wedge F_2$ ,  $F_1 \vee F_2$ , and  $F_1 \rightarrow F_2$  are formulas where  $i \in \{\mathbf{Tim}, \mathbf{Sit}\}$ .

The existential predicate formula  $E(t)$  asserts the existence of entities denoted by a term  $t$  in the domain of discourse. The modal formulas are constructed by the modal operators  $\blacksquare, \blacklozenge$  (any world),  $\Box_{\mathbf{Tim}}, \Diamond_{\mathbf{Tim}}$  (temporal), and  $\Box_{\mathbf{Sit}}, \Diamond_{\mathbf{Sit}}$  (situational). To axiomatize rigidity and dependencies with individual existence, the modality  $\blacksquare F$  and  $\Box_i F$  asserts that  $F$  holds for any accessible world whenever individuals exist. For example, the sorted modal formula

$$\Box_{\mathbf{Tim}} p_{male}(bob_{person})$$

implies that for any time accessible from a world, Bob is a male person *as long as he exists*.

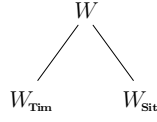


Figure 2: Types of possible worlds

### 3.2 Semantics

We define the semantics for a sorted first-order modal language  $\mathcal{L}_\Sigma$  as follows:

**Definition 12 (Sorted  $\Sigma$ -Structures)** *Let  $\Sigma$  be a sorted signature. A sorted  $\Sigma$ -structure  $M$  is a tuple  $(W, w_0, R, R', U, I)$  such that*

1.  $W$  is a superset of  $\bigcup_{1 \leq i \leq n} W_i$  where  $W_i$  is a non-empty set of worlds and  $W_i \cap W_j = \emptyset$  ( $i \neq j$ );
2.  $R = (R_1, \dots, R_n)$  where  $R_i$  is a subset of  $W \times W_i$ ;
3.  $R'$  is a superset of  $R_1 \cup \dots \cup R_n$ ;
4.  $U$  is a superset of  $\bigcup_{w \in W} U_w$  where  $U_w$  is the set of individuals in world  $w$ <sup>2</sup>; and
5.  $I = \{I_w \mid w \in W\}$  is the set of interpretation functions  $I_w$  for all worlds  $w \in W$  with the following conditions:
  - (a) if  $s \in T \cup S_A$ , then  $I_w(s) \subseteq U_w$ , (in particular, if  $s = \top$ , then  $I_w(s) = U_w$ ). In addition,  $I(s)$  is a superset of  $\bigcup_{w \in W} I_w(s)$  such that  $U_w \cap I(s) \subseteq I_w(s)$ <sup>3</sup>,
  - (b) if  $s_i \leq s_j$  with  $s_i, s_j \in T \cup S_A$ , then  $I_w(s_i) \subseteq I_w(s_j)$ ,
  - (c) if  $c \in C$  and  $c: \rightarrow \tau \in \Omega$ , then  $I_w(c) \in I(\tau)$ ,
  - (d) if  $f \in F_n$  and  $f: \tau_1 \times \dots \times \tau_n \rightarrow \tau \in \Omega$ , then  $I_w(f): I(\tau_1) \times \dots \times I(\tau_n) \rightarrow I(\tau)$ ,
  - (e) if  $p \in P_n$  and  $p: s_1 \times \dots \times s_n \in \Omega$ , then  $I_w(p) \subseteq I_w(s_1) \times \dots \times I_w(s_n)$  (in particular, if  $p_s \in P_{T \cup S_A}$  and  $p_s: \tau \in \Omega$ , then  $I_w(p_s) \subseteq I_w(\tau)$ ), and
  - (f) if  $p \in P_{non}$  and  $p: undef \in \Omega$ , then  $I_w(p) \subseteq U_w \cup \Delta_w$  where  $\Delta_w$  is an uncountably infinite set<sup>4</sup>.

We add the definition  $I_w(\prec) \subseteq (U_w \cup \Delta_w) \times (U_w \cup \Delta_w)$  to sorted  $\Sigma$ -structures in order to interpret the part-of relation  $\prec$ :  $undef \times undef \in \Omega$ .

In the semantics, we introduce an arbitrary number  $n$  of world sets  $W_1, \dots, W_n$  and accessibility relations  $R_1, \dots, R_n$  in order to generally define  $n$  modal operators  $\Box_1, \dots, \Box_n$  and  $\Diamond_1, \dots, \Diamond_n$ . Each of the modal operators  $\Box_i$  and  $\Diamond_i$  are defined over the pair of the set  $W$  of any worlds and a set  $W_i$  of worlds. In the following definition using Definitions 3 - 5 (of time and situation dependencies), accessibility relations from worlds to times and

<sup>2</sup>Each world can have a different domain (possibly  $U_{w_1} \neq U_{w_2}$ ).

<sup>3</sup>If an individual in  $I(s)$  exists in a world  $w$ , then it must belong to the interpretation  $I_w(s)$  in  $w$ . That is,  $I(s)$  may be constructed by  $\bigcup_{w \in W} I_w(s)$  and individuals non-existing in any world.

<sup>4</sup>In order to interpret each non-sortal predicate, an uncountably infinite set is introduced. For example, the non-sortal property *water* has to be characterized by an uncountable infinite set of elements corresponding to the substance of water.

situations correspond to the two specific modal operators  $\Box_{\mathbf{Tim}}, \Box_{\mathbf{Sit}}$  (and  $\Diamond_{\mathbf{Tim}}, \Diamond_{\mathbf{Sit}}$ ). In Kripke semantics, we employ three sets  $W$ ,  $W_{\mathbf{Tim}}$ , and  $W_{\mathbf{Sit}}$  of worlds and their accessibility relations  $R'$ ,  $R_{\mathbf{Tim}}$ , and  $R_{\mathbf{Sit}}$ , defining the rigidity of *any worlds* (in  $W$ ) and the dependencies of *time* (in  $W_{\mathbf{Tim}}$ ) and *situation* (in  $W_{\mathbf{Sit}}$ ). The three types of possible worlds construct the sort hierarchy shown in Figure 2. Times and situations can be called worlds generally because their types  $W_{\mathbf{Tim}}$  and  $W_{\mathbf{Sit}}$  are subsorts of  $W$ . In this paper, other specific possible worlds are not introduced since we focus on the dependencies of times and situations. For instance, beliefs are possible worlds that are neither times nor situations.

The modalities in sorted  $\Sigma$ -structures interpret the rigidity and time and situation dependencies as the semantic differences among types  $\tau$  and anti-rigid sorts  $\sigma$ , as given below.

**Definition 13 (Rigidity, Time, and Situation)** *Let  $\Sigma$  be a sorted signature. A sorted  $\Sigma$ -structure with rigidity and time and situation dependencies (called sorted  $\Sigma^+$ -structure) is a sorted  $\Sigma$ -structure  $M = (W, w_0, R, R', U, I)$  such that*

*(rigidity)*

1.  $R' \supseteq R_{\mathbf{Tim}} \cup R_{\mathbf{Sit}}$  is reflexive and transitive,
2. Definition 1 (in Section 2),
3. for every generic predicate  $p \in P_{non}$ , if  $d \in I_w(p)$  and  $(w, w') \in R'$ , then  $d \in U_{w'} \cup \Delta_{w'}$  implies  $d \in I_{w'}(p)$ ,

*(time and situation dependencies)*

4.  $W$  is a superset of  $W_{\mathbf{Tim}} \cup W_{\mathbf{Sit}}$  where  $W_{\mathbf{Tim}}$  is the set of times and  $W_{\mathbf{Sit}}$  is the set of situations ( $W_{\mathbf{Tim}} \cap W_{\mathbf{Sit}} = \emptyset$ ),
5.  $R = (R_{\mathbf{Tim}}, R_{\mathbf{Sit}})$  where  $R_{\mathbf{Tim}} \subseteq W \times W_{\mathbf{Tim}}$  is reflexive and transitive over  $W_{\mathbf{Tim}} \times W_{\mathbf{Tim}}$  and  $R_{\mathbf{Sit}} \subseteq W \times W_{\mathbf{Sit}}$  is reflexive and transitive over  $W_{\mathbf{Sit}} \times W_{\mathbf{Sit}}$ ,
6. Definitions 3-5 (in Section 2),

*(part-of relation)*

7. Definition 6 (in Section 2).

Although there are various time representations such as time points and time periods in a linear structure or a branching structure in the literature of temporal reasoning, we do not select such a specific time representation. Since the rigidity and dependencies are specified by the interpretation of properties over worlds, times, and situations, we prefer to treat worlds, times, and situations as arbitrary elements, unlike time points and time periods. Therefore, in our formalization, the accessibility relation  $R_{\mathbf{Tim}}$  defines a general relation over times ( $W_{\mathbf{Tim}} \times W_{\mathbf{Tim}}$ ) as follows:

- $tmR_{\mathbf{Tim}}tm'$ :  $tm$  is related to a time  $tm'$  if  $tm'$  is a time accessible from a time  $tm$ .

Based on this direction, the accessibility relation  $R_{\mathbf{Sit}}$  defines a general relation over situations ( $W_{\mathbf{Sit}} \times W_{\mathbf{Sit}}$ ) as follows:

- $stR_{\mathbf{Tim}}st'$ :  $st$  is related to a situation  $st'$  if  $st'$  is a situation accessible from a situation  $st$ .

More generally, the accessibility relation  $R'$  over  $W \times W$  is regarded as a general relation over any worlds. In the formal definition of modalities, it is useful to consider that the accessibility relations  $R'$ ,  $R_{\text{Tim}}$ , and  $R_{\text{Sit}}$  are reflexive and transitive over the same type of worlds, i.e., S4-like modalities. This is because the S4 modalities intuitively interpret properties over times and situations, e.g., if  $A$  is true anytime, then  $A$  is true now (reflexivity), and if  $A$  is true anytime, then it is true anytime that  $A$  is true anytime (transitivity). Moreover,  $R'$  supports the same modality because we define times and situations as subsets of the set of general worlds. This option enables us to combine a consistency checking algorithm with our formal ontology such that time and situation dependencies are defined without losing the generality of rigidity. In other words, the S4 modalities lead to the fact that if a property is generally rigid, then it is temporally stable and situationally stable in the semantics. This is the reason why we do not define the semantics over the Cartesian product of  $W_{\text{Tim}}$  and  $W_{\text{Sit}}$  (or  $W_{\text{Sit}}$  and  $W_{\text{Tim}}$ ).

We characterize the time and situation dependencies of properties by using the accessibility relations  $R_{\text{Tim}}$  and  $R_{\text{Sit}}$  over the pair of the set  $W$  of worlds and the set  $W_{\text{Tim}}$  of times and over the pair of the set  $W$  of worlds and the set  $W_{\text{Sit}}$  of situations, respectively<sup>5</sup>. Moreover, for the rigidity over any worlds, the accessibility relation  $R'$  is defined as a superset of  $R_{\text{Tim}}$  and  $R_{\text{Sit}}$ . For the dependencies of time and situation, the intuitive meaning of the accessibility relations  $R_{\text{Tim}}$  and  $R_{\text{Sit}}$  is informally explained as follows.

- $tmR_{\text{Sit}}st$ :  $st$  is a situation within a time  $tm$  if  $st$  is accessible from  $tm$ .
- $stR_{\text{Tim}}tm$ :  $tm$  is a time in a situation  $st$  if  $tm$  is accessible from  $st$ .

These relations will be used to define the meaning of the situational operators  $\Box_{\text{Sit}}$  and  $\Diamond_{\text{Sit}}$  and the temporal operators  $\Box_{\text{Tim}}$  and  $\Diamond_{\text{Tim}}$ .

Further, there exist the correspondences between sorts and their sort predicates (based on Definition 2). If a type  $\tau$  is situation dependent, then this type and its type predicate are *extensible* with  $I_w(\tau) \subsetneq I_w(p_\tau)$  (in addition,  $I_w(p_\tau) \subseteq I_w(p_{\tau'})$  if  $\tau \leq \tau'$  and  $\tau, \tau'$  are extensible). Any other sort and its sort predicate are *inextensible* with  $I_w(s) = I_w(p_s)$ . For every extensible type predicate  $p_\tau$ , we assume that there exists an anti-rigid sort  $\sigma$  as the role of type  $\tau$  with  $\sigma \leq \tau$  such that  $I_w(\sigma) = I_w(p_\tau) \setminus I_w(\tau)$ . For instance, the anti-rigid sort *temporary\_weapon* is a role of the type *weapon*. A sort predicate formula  $p_s(t)$  is simply denoted by  $s(t)$  if it will not cause confusion. Unlike types, every anti-rigid sort is primarily anti-rigid.

In order to define the denotation of terms, we introduce the set  $C_U$  of new constants  $\bar{d}$  for individuals  $d$  in  $U$  where every new constant is interpreted by itself. In what follows, we will adopt a sorted first-order modal language  $\mathcal{L}_\Sigma$  extended by adding the set  $C_U$  of new constants.

**Definition 14** *Let  $M = (W, w_0, R, R', U, I)$  be a sorted  $\Sigma^+$ -structure. The denotation  $\llbracket \cdot \rrbracket_w : \mathcal{T}_\top \rightarrow U_w$  is defined by the following rules:*

1.  $\llbracket c_\tau \rrbracket_w = I_w(c)$  for  $c \in C$  where  $c : \rightarrow \tau \in \Omega$ ,
2.  $\llbracket \bar{d} \rrbracket_w = d$  for  $\bar{d} \in C_U$ , and

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<sup>5</sup>The reverse relations  $W_{\text{Tim}} \times W$  and  $W_{\text{Sit}} \times W$  and others  $W_{\text{Tim}} \times W_{\text{Sit}}$  and  $W_{\text{Sit}} \times W_{\text{Tim}}$  are not needed because the three types of worlds accessible from any world sufficiently define the three modal operators, e.g.,  $W_{\text{Tim}} \times W$  and  $W_{\text{Tim}} \times W_{\text{Sit}}$  are defined by  $W \times W$  and  $W \times W_{\text{Sit}}$ , respectively.

3.  $\llbracket f_{\tau^*,\tau}(t_1, \dots, t_n) \rrbracket_w = I_w(f)(\llbracket t_1 \rrbracket_w, \dots, \llbracket t_n \rrbracket_w)$  for  $f \in F_n$  where  $f: \tau_1 \times \dots \times \tau_n \rightarrow \tau \in \Omega$ .

Note that  $\mathcal{T}_\top$  (the domain of  $\llbracket \cdot \rrbracket_w$ ) denotes the set of terms of all sorts since  $\top$  is the greatest type and  $\mathcal{T}_s$  denotes the set of terms of sort  $s$  and all subsorts of it. A closed formula is a formula without free variables. We define the set of subterms of a term  $t$  as follows:

1. if  $t = c_\tau$ , then  $sub(t) = \{c_\tau\}$ .
2. if  $t = f_{\tau^*,\tau}(t_1, \dots, t_n): s$ , then  $sub(t) = \{f_{\tau^*,\tau}(t_1, \dots, t_n): s\} \cup sub(t_1) \cup \dots \cup sub(t_n)$ .

To define satisfiability of modal formulas, we deal with the existence of terms in each world. Let  $M = (W, w_0, R, R', U, I)$  be a sorted  $\Sigma^+$ -structure, let  $w \in W$ , and let  $\llbracket t \rrbracket_w$  be the denotation of a term  $t$  in  $w$ . The set  $Nex_w$  of formulas with terms non-existing in  $w$  is the smallest set such that

1.  $p(t_1, \dots, t_n) \in Nex_w$  iff for some ground term  $t \in sub(t_1) \cup \dots \cup sub(t_n)$ ,  $\llbracket t \rrbracket_w \notin U_w$ ;
2.  $\neg F$ ,  $(\forall x_s)F$ ,  $(\exists x_s)F \in Nex_w$  iff  $F \in Nex_w$ ;
3.  $\Box_i F$ ,  $\Diamond_i F$ ,  $\blacksquare F$ ,  $\blacklozenge F \notin Nex_w$ ;
4.  $F_1 \wedge F_2 \in Nex_w$  iff  $F_1 \in Nex_w$  or  $F_2 \in Nex_w$ ;
5.  $F_1 \vee F_2 \in Nex_w$  iff  $F_1 \in Nex_w$  and  $F_2 \in Nex_w$ ; and
6.  $F_1 \rightarrow F_2 \in Nex_w$  iff  $\neg F_1 \in Nex_w$  and  $F_2 \in Nex_w$ .

This set  $Nex_w$  is important for the interpretation of modality. In Definition 15, the modal formula  $\blacksquare F$  is satisfied in a world  $w$  if for any world  $w'$  accessible from  $w$ ,  $F$  is satisfied in  $w'$  ( $w' \models F$ ) or some ground terms in  $F$  do not exist in  $w'$  ( $F \in Nex_{w'}$ ).

**Definition 15 ( $\Sigma^+$ -Satisfiability Relation)** Let  $M = (W, w_0, R, R', U, I)$  be a  $\Sigma^+$ -structure, let  $F \in \mathcal{A} \cup \mathcal{G} \cup \mathcal{C}$ , and let  $w \in W$ . The  $\Sigma^+$ -satisfiability relation  $w \models F$  is defined inductively as follows:

1.  $w \models p(t_1, \dots, t_n)$  iff  $(\llbracket t_1 \rrbracket_w, \dots, \llbracket t_n \rrbracket_w) \in I_w(p)$ .
2.  $w \models E(t)$  iff there exists  $d \in U_w$  such that  $\llbracket t \rrbracket_w = d$ .
3.  $w \models \neg F$  iff  $w \not\models F$ .
4.  $w \models F_1 \wedge F_2$  iff  $w \models F_1$  and  $w \models F_2$ .
5.  $w \models F_1 \vee F_2$  iff  $w \models F_1$  or  $w \models F_2$ .
6.  $w \models F_1 \rightarrow F_2$  iff  $w \not\models F_1$  or  $w \models F_2$ .
7.  $w \models (\forall x_s)F$  iff for all  $d \in I_w(s)$ ,  $w \models F[x_s/\bar{d}]$ .
8.  $w \models (\exists x_s)F$  iff for some  $d \in I_w(s)$ ,  $w \models F[x_s/\bar{d}]$ .
9.  $w \models \Box_i F$  (resp.  $\blacksquare F$ ) iff for all  $w' \in W_i$  with  $(w, w') \in R_i$  (resp.  $R'$ ),  $w' \models F$  or  $F \in Nex_{w'}$ .

10.  $w \models \diamond_i F$  (resp.  $\blacklozenge F$ ) iff for some  $w' \in W_i$  with  $(w, w') \in R_i$  (resp.  $R'$ ),  $w' \models F$  and  $F \notin \text{Nex}_{w'}$ .

Let  $F$  be a formula. It is  $\Sigma^+$ -true in  $M$  if  $w_0 \models F$  ( $M$  is a  $\Sigma^+$ -model of  $F$ ). If  $F$  has a  $\Sigma^+$ -model, it is  $\Sigma^+$ -satisfiable, otherwise, it is  $\Sigma^+$ -unsatisfiable.  $F$  is  $\Sigma^+$ -valid if every sorted  $\Sigma^+$ -structure is a  $\Sigma^+$ -model of  $F$ .

In the semantics, for each sort  $s \in T \cup S_A$ ,  $I(s)$  is the interpretation of sort  $s$  without individual existence and worlds, and  $I_w(s)$  is the interpretation of sort  $s$  existing in a world  $w$ . For example,  $I(\text{person})$  implies “all the persons some of which may not exist in any world” and  $I_w(\text{person})$  implies “all the persons in a world  $w$ .” By the  $\Sigma^+$ -satisfiability relation,  $\forall x_s$  and  $\exists x_s$  respectively express universal and existential quantifications over the individuals of  $I_w(s)$  in a world  $w$  (but some of the individuals in  $w$  may not exist in another world  $w'$ ). For example, consider

$$w \models (\exists x_s)(p_1(x_s) \wedge \blacksquare \neg p_2(x_s))$$

where  $x_s$  is existentially quantified over individuals in the world  $w$ . The subformula  $\blacksquare \neg p_2(x_s)$  indicates that for every world  $w'$  accessible from  $w$ , the quantified value of  $x_s$  in  $w$  does not belong to the predicate  $p_2$ , precisely, it exists as a non-member of  $p_2$  or does not exist in  $w'$ .

### 3.3 Axioms

**Proposition 1** *Let  $p$  be an inextensible type predicate  $p_{\tau'}$  with  $p_{\tau'} : \tau \in \Omega$  or generic predicate (in this case,  $\tau = \top$ ),  $p_{\sigma}$  be an anti-rigid sort predicate with  $p_{\sigma} : \tau \in \Omega$ , and  $q$  be an anti-sortal predicate. The following axioms are  $\Sigma^+$ -valid.*

1. **Rigid predicate axiom:**  
 $(\forall x_{\tau})(p(x_{\tau}) \rightarrow \blacksquare p(x_{\tau}))$
2. **Anti-rigid predicate axiom:**  
 $(\forall x_{\tau})(p_{\sigma}(x_{\tau}) \rightarrow \blacklozenge(\neg p_{\sigma}(x_{\tau})))$
3. **Time dependency axiom:**  
 $\square_{\mathbf{Tim}}(\forall x_{\tau})(p_{\sigma}(x_{\tau}) \rightarrow \diamond_{\mathbf{Tim}}(\neg p_{\sigma}(x_{\tau})))$   
 $\square_{\mathbf{Tim}}(\forall x_{\tau})(p_{\sigma}(x_{\tau}) \rightarrow \square_{\mathbf{Sit}} p_{\sigma}(x_{\tau}))$
4. **Situation dependency axiom:**  
 $\square_{\mathbf{Sit}}(\forall x_{\tau})(p_{\sigma}(x_{\tau}) \rightarrow \diamond_{\mathbf{Sit}}(\neg p_{\sigma}(x_{\tau})))$   
 $\square_{\mathbf{Sit}}(\forall x_{\tau})(p_{\sigma}(x_{\tau}) \rightarrow \square_{\mathbf{Tim}} p_{\sigma}(x_{\tau}))$
5. **Time-situation dependency axiom:**  
 $\square_{\mathbf{Sit}}(\forall x_{\tau})(p_{\sigma}(x_{\tau}) \rightarrow \diamond_{\mathbf{Sit}}(\neg p_{\sigma}(x_{\tau})))$   
 $\square_{\mathbf{Sit}}(\forall x_{\tau})(p_{\sigma}(x_{\tau}) \rightarrow (\diamond_{\mathbf{Tim}} p_{\sigma}(x_{\tau}) \wedge \diamond_{\mathbf{Tim}}(\neg p_{\sigma}(x_{\tau}))))$
6. **Part-of relation axiom:**  
 $(\forall x)(\forall y)(q(x) \wedge (y \prec x) \rightarrow q(y))$

We denote the set of axioms by  $\mathcal{AL}_{\Sigma}$ . Our proposed sorted modal logic takes account of the notion of individual existence, and therefore, these axioms reflect the intension of rigidity and time and situation dependencies. In other words, they correspond to the semantic

constraints in Definitions 2 - 6. The first axiom implies that if a rigid predicate holds, then it holds in any world as long as  $x$  exists. The second axiom implies that if an anti-rigid predicate holds, then there exists another world wherein it does not hold. The time dependence axiom indicates the anti-rigidity in a particular time and the rigidity in any situation within the time that the property holds. Similarly, the situation dependency axiom implies the anti-rigidity in a particular situation and the rigidity in any time within the situation that the property holds. The time-situation dependency axiom expresses the anti-rigidity in a situation, and if the property holds in a situation, then the property holds in a time but not in another time. These axioms become valid in the sorted  $\Sigma^+$ -structure satisfying the semantic conditions of rigidity and dependencies.

Some researchers may consider that the axioms appear to be considerably clearer than those in Definitions 2 - 6 and that they should be defined before the definitions. However, the modal formulas in axioms cannot precisely imply our property classification without the definitions. This is because standard modal logic does not support the individual existence of modal operators in the Kripke semantics. That is, in order to support it, the modal formula  $A$  must assert the fact that  $A$  is true in any world accessible from a world as long as its included individual exists. Hence, we should first specify the semantic constraints of rigidity and time and situation dependencies including the individual existence. Second, the axioms with individual existence can be introduced through modal formulas in the refined sorted modal logic.

**Example 1** Let  $p_{apple}$  be an inextensible type predicate and  $p_{nov\_teacher}$  be an anti-rigid sort predicate (time-situation dependent) where  $p_{apple} : fruit$  and  $p_{nov\_teacher} : person$  in  $\Omega$ . Then, the two sorted modal formulas  $p_{apple}(c_{fruit}) \rightarrow \blacksquare p_{apple}(c_{fruit})$  and

$$\begin{aligned} \square_{\text{Sit}}(p_{nov\_teacher}(john_{person}) \rightarrow (\diamond_{\text{Tim}} p_{nov\_teacher}(john_{person}) \wedge \\ \diamond_{\text{Tim}} \neg p_{nov\_teacher}(john_{person}))) \end{aligned}$$

are  $\Sigma^+$ -valid. The subformula  $\blacksquare p_{apple}(c_{fruit})$  expresses rigidity with *individual existence*. This implies “ $c_{fruit}$  is an apple in any world as long as it exists,” (for any accessible world  $w \in W$ , if  $\llbracket c_{fruit} \rrbracket_w \in U_w$  (it exists), then  $\llbracket c_{fruit} \rrbracket_w \in I_w(p_{apple})$ ), but this does not imply “ $c_{fruit}$  is an apple forever” or “ $c_{fruit}$  exists forever.” Moreover, the subformula  $\diamond_{\text{Tim}} \neg p_{nov\_teacher}(john_{person})$  indicates “there is a time where  $john_{person}$  exists but is not a novice teacher.”

## 4 Tableau Calculus

In this section, we present a prefixed tableau calculus for the order-sorted modal logic.

### 4.1 Inference Rules

Let  $A$  be a closed formula in negation normal form (i.e., negation occurs only in front of an atomic formula) and  $S$  be a finite set of closed formulas in negation normal form. We define the annotated term  $t^n$  by annotating each constant symbol and function symbol with level  $n \in \mathbb{N}$  (e.g.,  $c_\tau^n$  and  $f_{\tau_1, \tau_2, \tau}^n(x_{\tau_1}, c_{\tau_2}^n)$ ). The annotated set  $S^n$  and formula  $A^n$  are obtained as follows: If  $S = \{A_1, \dots, A_m\}$  then  $S^n = \{A_1^n, \dots, A_m^n\}$ . If  $A_i = p(t_1, \dots, t_m)$  then  $A_i^n = p(t_1^n, \dots, t_m^n)$ , and otherwise,  $A_i^n = A_i$ . The annotated term  $t^n$  implies that it exists in the world corresponding to level  $n$ . Each node in a tableau is labeled with a prefixed

formula set  $(i, n): S$  where  $i \in \{W, Tim, Sit\}$  and  $n \in \mathbb{N}$ . The initial tableau for  $S$  is the single node  $(W, 1): (S^+)^1$  where  $S^+$  is the smallest superset of  $S$  obtained by adding  $\blacksquare F$  for all axioms  $F$  in  $\mathcal{A}_{\mathcal{L}_\Sigma}$ . The initial tableau  $(W, 1): (S^+)^1$  plays the key role in deciding  $\Sigma^+$ -satisfiability for  $S$  since it includes the formulas  $\blacksquare F$  for all axioms  $F$  in  $\mathcal{A}_{\mathcal{L}_\Sigma}$ . The axioms characterize the meta-features of properties, and the attached operator  $\blacksquare F$  validates the axioms in any world by applications of  $\pi_j$ -rule,  $\pi_{i \rightarrow j}$ -rule, and  $\pi_W$ -rule.

The tableau calculus contains conjunction and disjunction rules, existential predicate rules, modal operator rules, sorted quantifier rules, and sort predicate rules. The pair  $(i, n)$  of labels  $i$  and  $n$  denotes the type of worlds and the world itself. A ground term  $t$  is of level  $n$  if the annotated term  $t^n$  occurs in an ancestor. Let  $i \in \{W, Tim, Sit\}$ ,  $j \in \{Tim, Sit\}$ , let  $t$  be any ground term with level  $n$ , and let comma be the union of sets (i.e.,  $S_1, S_2 = S_1 \cup S_2$ ,  $A, S = \{A\} \cup S$ , and  $A, B = \{A\} \cup \{B\}$ ).

<b>Conjunction and disjunction rules</b>
$\frac{(i, n): A \wedge B, S}{(i, n): A^n, B^n, S} (\alpha) \quad \frac{(i, n): A \vee B, S}{(i, n): A^n, S \quad (i, n): B^n, S} (\beta)$

The conjunction and disjunction rules are based on the standard tableau rules. It can be seen that if a formula is decomposed into atomic formulas, the ground terms in the atomic formulas are labeled with the natural number  $n$ , as  $A^n$ . This indicates that the included terms exist in world  $n$ , and they are referred to as terms with level  $n$ .

For our proposed sorted modal logic, we have to handle the multi-modal operators with individual existence. To do this, modal formulas are prefixed by a pair  $(i, n)$  of the type of worlds  $i$  and a natural number  $n$ . For example, the prefixed formula  $(Tim, 3): p(c^3)$  indicates that the constant  $c$  is an individual of the time world numbered by 3 and  $p(c)$  is true in the world. In  $\alpha$ -rule and  $\beta$ -rule, the decomposed formulas  $A$  and  $B$  are annotated with level  $n$  (such as  $A^n$  and  $B^n$ ) since they may be atomic formulas. For example, if  $p(t) \wedge F$  is decomposed to  $p(t)$  and  $F$  by  $\alpha$ -rule, then we obtain the annotated atomic formula  $p(t^n)$ .

<b>Existential predicate rules</b>
$\frac{(i, n): \neg E(t), S}{(i, n): \perp, \neg E(t), S} (E) \quad \frac{(i, n): S}{(i, n): E(a_n^n), p_s(a_n^n), S} (I)$

These rules are introduced to cover the existential predicate. In an  $E$ -rule, if the term  $t$  has level  $n$ , it derives a contradiction because  $t$  must exist in world  $n$ . In an  $I$ -rule, a new constant  $a$  is introduced as a dummy constant of sort  $s$ . This guarantees the existence of an individual in every world. Then,  $a_n$  is the dummy constant for level  $n$  such that  $sort(a_n) \leq s$  for all sorts  $s \in T \cup S_A$  ( $a_n^n$  is the annotated term of  $a_n$  with level  $n$ ). Furthermore, if  $a_n$  is a dummy constant, then  $I_w(a_n) \in I(s)$  for all sorts  $s \in T \cup S_A$  in every sorted  $\Sigma^+$ -structure. By an application of  $I$ -rule,  $a_n$  is introduced as a ground term with level  $n$ . The dummy constant for each level is used to guarantee the non-empty domain of each world.

In the modal operation rules,  $*S$  denotes  $\{*F \mid F \in S\}$  for  $* \in \{\blacksquare, \Box_i\}$  (possibly  $*S = \emptyset$ ). Let  $\mathcal{T}_0$  be the set of ground terms. The translation function  $\mathcal{E}$  is defined by  $\mathcal{E}(p(t_1, \dots, t_n)) = \emptyset$  if  $\{t_1, \dots, t_n\} \not\subseteq \mathcal{T}_0$ , otherwise  $\mathcal{E}(p(t_1, \dots, t_n)) = E(t_1) \wedge \dots \wedge E(t_n)$ . This translation generates the conjunction of ground terms occurring in an atomic formula. For any formula  $F$ , the function  $\mathcal{E}$  can be expanded as follows: (i)  $\mathcal{E}(*F) = \mathcal{E}(F)$  for every  $* \in$

$\{\neg, \forall x_s, \exists x_s\}$ , (ii)  $\mathcal{E}(*F) = \emptyset$  for every  $* \in \{\Box_i, \Diamond_i, \blacksquare, \blacklozenge\}$ , (iii)  $\mathcal{E}(F_1 \wedge F_2) = \mathcal{E}(F_1) \wedge \mathcal{E}(F_2)$ , and (iv)  $\mathcal{E}(F_1 \vee F_2) = \mathcal{E}(F_1) \vee \mathcal{E}(F_2)$ . Moreover, we define  $S \vee \neg\mathcal{E}(S) = \{F \vee \neg\mathcal{E}(F) \mid F \in S\}$ .

<b>Modal operator rules</b>	
$\frac{(j, n): \Box_j A, S}{(j, n): A \vee \neg\mathcal{E}(A), \Box_j A, S} (\nu_j)$	$\frac{(j, n): \Diamond_j A, \Box_j S \cup \blacksquare S, S'}{(j, n+1): A \wedge \mathcal{E}(A), S \vee \neg\mathcal{E}(S), \Box_j S \cup \blacksquare S} (\pi_j)$
$\frac{(i, n): \Diamond_j A, \Box_j S \cup \blacksquare S, S'}{(j, n+1): A \wedge \mathcal{E}(A), S \vee \neg\mathcal{E}(S), \blacksquare S} (\pi_{i \rightarrow j})$	$\frac{(i, n): \blacklozenge A, \blacksquare S, S'}{(W, n+1): A \wedge \mathcal{E}(A), S \vee \neg\mathcal{E}(S), \blacksquare S} (\pi_W)$
$\frac{(i, n): \blacksquare A, S}{(i, n): A \vee \neg\mathcal{E}(A), \Box_{\mathbf{Tim}} A, \Box_{\mathbf{Sit}} A, \blacksquare A, S} (\blacksquare\Box)$	$\frac{(i, n): \Diamond_j A, S}{(i, n): \blacklozenge A, \Diamond_j A, S} (\Diamond\blacklozenge)$

The  $\nu_j$ -rule derives the disjunction of the formula  $A$  and the negation  $\neg\mathcal{E}(A)$  of the existential predicate formula. In other words, the disjunction implies that the formula  $A$  is true or some ground terms in  $A$  do not exist.

The  $\pi_j$ -rule,  $\pi_{i \rightarrow j}$ -rule, and  $\pi_W$ -rule derive the conjunction of the formula  $A$  and the existence  $\mathcal{E}(A)$  of the ground terms in  $A$  and these increase the natural number  $n$  (by the annotation  $(j, n+1)$  in the conclusion), since the modal operators  $\Diamond_j$  and  $\blacklozenge$  imply that there exists a world  $n+1$  accessible from  $n$ . In  $\pi_{i \rightarrow j}$ -rule,  $i \neq j$ , in  $\pi_j$ -rule and  $\pi_{i \rightarrow j}$ -rule,  $S'$  is a set of closed formulas without the forms  $\blacksquare F$  and  $\Box_j F$ , and in  $\pi_W$ -rule,  $S'$  is a set of closed formulas without the form  $\blacksquare F$ . The  $\pi_j$ -rule does not modify the type of worlds but the  $\pi_{i \rightarrow j}$  rule modifies the type of worlds  $i$  into  $j$  (by replacing the annotation  $(i, n)$  with the annotation  $(j, n+1)$ ). Further, when the  $\pi_W$ -rule is applied to a black diamond formula, the type of worlds is labeled as  $W$  (by the annotation  $(W, n+1)$  in the conclusion), denoting any type of worlds.

Additionally,  $\blacksquare\Box$ -rule and  $\Diamond\blacklozenge$ -rule are introduced by supporting the fact that the possible worlds of time and situation are a subset of the set of worlds. Cialdea-Mayer and Cerrito's prefixed tableau calculus uses formulas labeled with a natural number  $n$ . In contrary, our calculus is complicatedly extended to handle existential predicate formulas and processes formulas prefixed by the pair  $(i, n)$  of the type of worlds  $i$  and a natural number  $n$  because of the multi-modal operators with individual existence.

The following sorted quantifier rules and sort predicate rules are applied to the quantifiers of typed variables and anti-rigid sorted variables.

<b>Sorted quantifier rules</b>	
$\frac{(i, n): \forall x_\tau A, S}{(i, n): A[x_\tau/t]^n, \forall x_\tau A, S} (\gamma_\tau)$	$\frac{(i, n): p_{s'}(t^n), \forall x_s A, S}{(i, n): p_{s'}(t^n), A[x_s/t]^n, \forall x_s A, S} (\gamma_s)$
$\frac{(i, n): \exists x_\tau A, S}{(i, n): E(c_\tau^n), A[x_\tau/c_\tau]^n, \exists x_\tau A, S} (\delta_\tau)$	$\frac{(i, n): \exists x_\sigma A, S}{(i, n): p_\sigma(c_\tau^n), A[x_\sigma/c_\tau]^n, \exists x_\sigma A, S} (\delta_\sigma)$

In  $\gamma_\tau$ -rule,  $sort(t) \leq \tau$ , in  $\gamma_s$ -rule,  $s' \leq s$  and if  $s$  is extensible, then  $p_{s'}$  is an anti-rigid sort predicate, in  $\delta_\tau$ -rule,  $c_\tau$  is a constant not in  $\{\exists x_\tau A\} \cup S$ , and in  $\delta_\sigma$ -rule,  $c_\tau$  is a constant not in  $\{\exists x_\sigma A\} \cup S$  where  $p_\sigma: \tau \in \Omega$ . Then,  $\gamma_\tau$ - and  $\delta_\tau$ -rules are used for typed variables,  $\delta_s$ -rule is used for anti-rigid sorted variables, and the  $\gamma_s$ -rule is used for both typed and anti-rigid

$$\begin{array}{c}
\frac{(W, 1): \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \wedge \blacklozenge(\neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}})), \blacksquare \forall x_{\text{fruit}}(\mathcal{P}_{\text{apple}}(x_{\text{fruit}}) \rightarrow \blacksquare \mathcal{P}_{\text{apple}}(x_{\text{fruit}}))}{(W, 1): \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \wedge \blacklozenge(\neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}})), \forall x_{\text{fruit}}(\mathcal{P}_{\text{apple}}(x_{\text{fruit}}) \rightarrow \blacksquare \mathcal{P}_{\text{apple}}(x_{\text{fruit}}))} (\blacksquare \square) \\
\frac{(W, 1): \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \wedge \blacklozenge(\neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}})), \forall x_{\text{fruit}}(\mathcal{P}_{\text{apple}}(x_{\text{fruit}}) \rightarrow \blacksquare \mathcal{P}_{\text{apple}}(x_{\text{fruit}}))}{(W, 1): \mathcal{P}_{\text{apple}}(c_{\text{fruit}}^1), \blacklozenge(\neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}})), \forall x_{\text{fruit}}(\mathcal{P}_{\text{apple}}(x_{\text{fruit}}) \rightarrow \blacksquare \mathcal{P}_{\text{apple}}(x_{\text{fruit}}))} (\alpha) \\
\frac{(W, 1): \mathcal{P}_{\text{apple}}(c_{\text{fruit}}^1), \blacklozenge(\neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}})), \forall x_{\text{fruit}}(\mathcal{P}_{\text{apple}}(x_{\text{fruit}}) \rightarrow \blacksquare \mathcal{P}_{\text{apple}}(x_{\text{fruit}}))}{(W, 1): \mathcal{P}_{\text{apple}}(c_{\text{fruit}}^1), \blacklozenge(\neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}})), \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \rightarrow \blacksquare \mathcal{P}_{\text{apple}}(c_{\text{fruit}})} (\gamma\tau) \\
\hline
(W, 1): \mathcal{P}_{\text{apple}}(c_{\text{fruit}}^1), \neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \quad (W, 1): \mathcal{P}_{\text{apple}}(c_{\text{fruit}}^1), \blacklozenge(\neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}})), \blacksquare \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \quad (\beta) \\
\frac{(W, 2): \neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \wedge E(c_{\text{fruit}}), \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \vee \neg E(c_{\text{fruit}})}{(W, 2): \neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}}), E(c_{\text{fruit}}^2), \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \vee \neg E(c_{\text{fruit}})} (\pi_W) \\
\frac{(W, 2): \neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}}), E(c_{\text{fruit}}^2), \mathcal{P}_{\text{apple}}(c_{\text{fruit}}) \vee \neg E(c_{\text{fruit}})}{(W, 2): \neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}}), \mathcal{P}_{\text{apple}}(c_{\text{fruit}}^2)} (\alpha) \\
\frac{(W, 2): \neg \mathcal{P}_{\text{apple}}(c_{\text{fruit}}), \mathcal{P}_{\text{apple}}(c_{\text{fruit}}^2) \quad (W, 2): \neg E(c_{\text{fruit}})}{(W, 2): \perp} (\beta) \\
(E)
\end{array}$$

Figure 3: A proof of satisfiability

sorted variables.

The sort predicate rules derive the type and sort predicate formulas from a subsort relation. If the type of a term  $t$  is a subtype of  $\tau$ , then this type predicate formula is true. Moreover, if  $s$  is a subsort of  $s'$ , then the sort predicate formula  $p_s$  derives the sort predicate formula  $p'_s$ .

Sort predicate rules	
$\frac{(i, n): S}{(i, n): p_\tau(t^n), S} (p_\tau)$	$\frac{(i, n): p_s(t^n), S}{(i, n): p_{s'}(t^n), p_s(t^n), S} (<)$

In  $p_\tau$ -rule,  $\text{sort}(t) \leq \tau$ , and in  $<$ -rule,  $s < s'$ .

A tableau rule is called *static* if it does not change the level  $n$  (i.e.,  $(i, n): S$  is expanded to  $(i, n): S'$  by an application of the rule), it is called *dynamic* otherwise (e.g.,  $\pi_j$ -rule,  $\pi_{i \rightarrow j}$ -rule, and  $\pi_W$ -rule are dynamic). The set of closed nodes in a tableau for  $(i, n): S$  is defined as follows:

- (i) if a node contains two complementary literals ( $\neg A$  and  $A^n$ ) or the clash symbol  $\perp$ , then it is closed, and
- (ii) if all the children of a node are closed, then it is closed.

A tableau is closed if the root is closed.

To test the satisfiability of any closed formula, the following proposition guarantees that any closed formula can be transformed into an equivalent one in negation normal form (i.e., negation occurs only in front of an atomic formula).  $FV(F)$  denotes the set of free variables occurring in a formula  $F$ . Let  $F_1$  and  $F_2$  be formulas where  $FV(F_1) \subseteq FV(F_2)$  and  $FV(F_1) \cup FV(F_2) = \{x_{s_1}^1, \dots, x_{s_n}^n\}$ .  $F_1 \simeq F_2$  is a semantic equivalence if for every sorted  $\Sigma^\perp$ -structure  $M = (W, w_0, R, R', U, I)$  and for every  $w \in W$ ,  $w \models F_1[x_{s_1}^1/\bar{d}_1, \dots, x_{s_n}^n/\bar{d}_n]$  if and only if  $w \models F_2[x_{s_1}^1/\bar{d}_1, \dots, x_{s_n}^n/\bar{d}_n]$ .

**Proposition 2** *Let  $F$ ,  $F_1$ , and  $F_2$  be formulas and let  $i \in \{\text{Tim}, \text{Sit}\}$ . The following*

$$\begin{array}{c}
\frac{(W, 1): \diamond_{\mathbf{Tim}} p_{\text{boy}}(\text{bob}_{\text{per}}) \wedge \square_{\mathbf{Tim}} (\forall x_{\text{per}}) \diamond_{\mathbf{Sit}} (\neg p_{\text{male}}(x_{\text{per}})), \blacksquare_{\mathbf{Tim}} (\forall y_{\text{per}}) (p_{\text{boy}}(y_{\text{per}}) \rightarrow \square_{\mathbf{Sit}} p_{\text{boy}}(y_{\text{per}}))}{(W, 1): \diamond_{\mathbf{Tim}} p_{\text{boy}}(\text{bob}_{\text{per}}) \wedge \square_{\mathbf{Tim}} (\forall x_{\text{per}}) \diamond_{\mathbf{Sit}} (\neg p_{\text{male}}(x_{\text{per}})), \square_{\mathbf{Tim}} (\forall y_{\text{per}}) (p_{\text{boy}}(y_{\text{per}}) \rightarrow \square_{\mathbf{Sit}} p_{\text{boy}}(y_{\text{per}}))} (\blacksquare_{\square}) \\
\frac{(W, 1): \diamond_{\mathbf{Tim}} p_{\text{boy}}(\text{bob}_{\text{per}}), \square_{\mathbf{Tim}} (\forall x_{\text{per}}) \diamond_{\mathbf{Sit}} (\neg p_{\text{male}}(x_{\text{per}})), \square_{\mathbf{Tim}} (\forall y_{\text{per}}) (p_{\text{boy}}(y_{\text{per}}) \rightarrow \square_{\mathbf{Sit}} p_{\text{boy}}(y_{\text{per}}))}{(Tim, 2): p_{\text{boy}}(\text{bob}_{\text{per}}) \wedge E(\text{bob}_{\text{per}}), (\forall x_{\text{per}}) \diamond_{\mathbf{Sit}} (\neg p_{\text{male}}(x_{\text{per}})), (\forall y_{\text{per}}) (p_{\text{boy}}(y_{\text{per}}) \rightarrow \square_{\mathbf{Sit}} p_{\text{boy}}(y_{\text{per}}))} (\alpha) \\
\frac{(Tim, 2): p_{\text{boy}}(\text{bob}_{\text{per}}^2), E(\text{bob}_{\text{per}}^2), (\forall x_{\text{per}}) \diamond_{\mathbf{Sit}} (\neg p_{\text{male}}(x_{\text{per}})), (\forall y_{\text{per}}) (p_{\text{boy}}(y_{\text{per}}) \rightarrow \square_{\mathbf{Sit}} p_{\text{boy}}(y_{\text{per}}))}{(Tim, 2): p_{\text{boy}}(\text{bob}_{\text{per}}^2), E(\text{bob}_{\text{per}}^2), \diamond_{\mathbf{Sit}} (\neg p_{\text{male}}(\text{bob}_{\text{per}})), (\forall y_{\text{per}}) (p_{\text{boy}}(y_{\text{per}}) \rightarrow \square_{\mathbf{Sit}} p_{\text{boy}}(y_{\text{per}}))} (\alpha) \\
\frac{(Tim, 2): p_{\text{boy}}(\text{bob}_{\text{per}}^2), E(\text{bob}_{\text{per}}^2), \diamond_{\mathbf{Sit}} (\neg p_{\text{male}}(\text{bob}_{\text{per}})), (\forall y_{\text{per}}) (p_{\text{boy}}(y_{\text{per}}) \rightarrow \square_{\mathbf{Sit}} p_{\text{boy}}(y_{\text{per}}))}{(Tim, 2): p_{\text{boy}}(\text{bob}_{\text{per}}^2), E(\text{bob}_{\text{per}}^2), \diamond_{\mathbf{Sit}} (\neg p_{\text{male}}(\text{bob}_{\text{per}})), p_{\text{boy}}(\text{bob}_{\text{per}}) \rightarrow \square_{\mathbf{Sit}} p_{\text{boy}}(\text{bob}_{\text{per}})} (\gamma_{\tau}) \\
\frac{(Tim, 2): p_{\text{boy}}(\text{bob}_{\text{per}}^2), \neg p_{\text{boy}}(\text{bob}_{\text{per}})}{(Tim, 2): p_{\text{boy}}(\text{bob}_{\text{per}}^2), E(\text{bob}_{\text{per}}^2), \diamond_{\mathbf{Sit}} (\neg p_{\text{male}}(\text{bob}_{\text{per}})), \square_{\mathbf{Sit}} p_{\text{boy}}(\text{bob}_{\text{per}})} (\pi_{i-j}) \\
\frac{(Sit, 3): \neg p_{\text{male}}(\text{bob}_{\text{per}}) \wedge E(\text{bob}_{\text{per}}), p_{\text{boy}}(\text{bob}_{\text{per}}) \vee \neg E(\text{bob}_{\text{per}})}{(Sit, 3): \neg p_{\text{male}}(\text{bob}_{\text{per}}), E(\text{bob}_{\text{per}}^3), p_{\text{boy}}(\text{bob}_{\text{per}}) \vee \neg E(\text{bob}_{\text{per}})} (\alpha) \\
\frac{(Sit, 3): \neg p_{\text{male}}(\text{bob}_{\text{per}}), p_{\text{boy}}(\text{bob}_{\text{per}}^3)}{(Sit, 3): \neg p_{\text{male}}(\text{bob}_{\text{per}}), \neg E(\text{bob}_{\text{per}})} (\beta) \\
\frac{(Sit, 3): \neg p_{\text{male}}(\text{bob}_{\text{per}}), p_{\text{male}}(\text{bob}_{\text{per}}^3)}{(Sit, 3): \neg p_{\text{male}}(\text{bob}_{\text{per}}), \perp} (<) (E)
\end{array}$$

Figure 4: A proof of satisfiability

*semantic equivalences hold:*

$$\begin{aligned}
\neg\neg F &\simeq F \\
\neg(F_1 \wedge F_2) &\simeq \neg F_1 \vee \neg F_2 \\
\neg(F_1 \vee F_2) &\simeq \neg F_1 \wedge \neg F_2 \\
\neg(F_1 \rightarrow F_2) &\simeq F_1 \wedge \neg F_2 \\
\neg(\forall x_s) F &\simeq (\exists x_s) \neg F \\
\neg(\exists x_s) F &\simeq (\forall x_s) \neg F \\
\neg\blacksquare F &\simeq \blacklozenge(\neg F) \\
\neg\blacklozenge F &\simeq \blacksquare(\neg F) \\
\neg\square_i F &\simeq \diamond_i(\neg F) \\
\neg\diamond_i F &\simeq \square_i(\neg F)
\end{aligned}$$

**Proof** Let  $M = (W, w_0, R, R', U, I)$  be any sorted  $\Sigma^+$ -structure and let  $w \in W$ . By Definition 15, the semantic equivalences can be proved as follows: Let  $FV(F) = \{x_{s_1}^1, \dots, x_{s_n}^n\}$  and let  $F' = F[x_{s_1}^1/\bar{d}_1, \dots, x_{s_n}^n/\bar{d}_n]$ .

$(\neg\neg F \simeq F)$   $w \models \neg\neg F'$  if and only if  $w \not\models \neg F'$  if and only if  $w \models F'$ .

$(\neg\blacklozenge F \simeq \blacksquare(\neg F))$   $w \models \neg\blacklozenge F'$  if and only if for all  $w' \in W$  with  $(w, w') \in R'$ ,  $w' \not\models F'$  or  $F' \in Nex_w$  if and only if for all  $w' \in W$  with  $(w, w') \in R'$ ,  $w' \models \neg F'$  or  $\neg F' \in Nex_w$  if and only if  $w \models \blacksquare \neg F'$ .

$(\neg\square_i F \simeq \diamond_i(\neg F))$   $w \models \neg\square_i F'$  if and only if for some  $w' \in W_i$  with  $(w, w') \in R_i$ ,  $w' \not\models F'$  and  $F' \notin Nex_w$  if and only if for some  $w' \in W_i$  with  $(w, w') \in R_i$ ,  $w' \models \neg F'$  and  $\neg F' \notin Nex_w$  if and only if  $w \models \diamond_i(\neg F')$ .

Similarly, the other cases can be shown. ■

## 4.2 Examples of Reasoning

Let us prove that the following sorted modal formula is  $\Sigma^+$ -valid by using the calculus.

$$F = p_{apple}(c_{fruit}) \rightarrow \blacksquare p_{apple}(c_{fruit})$$

(if  $c_{fruit}$  is an apple, then it is an apple in any world as long as it exists) where  $T = \{apple, fruit, \top\}$ ,  $S_A = \emptyset$ ,  $\leq$  is the transitive and reflexive closure of  $\{(apple, fruit)\}$ ,  $C = \{c\}$ ,  $P = \{p_{apple}, p_{fruit}, p_{\top}\}$ ,  $\Omega = \{c: \rightarrow fruit, p_{apple}: fruit, p_{fruit}: \top, p_{\top}: \top\}$ , and  $apple$  and  $fruit$  are inextensible in  $\Sigma$ .

In order to determine the validity of this formula, it is sufficient to check the satisfiability of its negation  $\neg F$ , i.e.,  $F$  is  $\Sigma^+$ -valid if and only if  $\neg F$  is  $\Sigma^+$ -unsatisfiable. By Proposition 2, the formula  $\neg F$  is transformed into an equivalent one in negation normal form as follows:

$$\begin{aligned} \neg(p_{apple}(c_{fruit}) \rightarrow \blacksquare p_{apple}(c_{fruit})) &\simeq p_{apple}(c_{fruit}) \wedge \neg \blacksquare p_{apple}(c_{fruit}) \\ &\simeq p_{apple}(c_{fruit}) \wedge \blacklozenge(\neg p_{apple}(c_{fruit})) \end{aligned}$$

Figure 3 illustrates a proof of testing the satisfiability of the formula  $\neg F$  where every tableau for  $S = \{\neg F\}$  is closed. This derives that the formula  $\neg F$  is  $\Sigma^+$ -unsatisfiable, and hence  $F$  is  $\Sigma^+$ -valid.

Furthermore, consider testing the validity of the following sorted modal formula:

$$F' = \blacklozenge_{\mathbf{Tim}} p_{boy}(bob_{person}) \rightarrow \blacklozenge_{\mathbf{Tim}}(\exists x_{person})(\Box_{\mathbf{Sit}} p_{male}(x_{person}))$$

(if Bob is a boy at a time, a person exists at a time who is male in any situation within the time) where  $T = \{person, male, animal, \top\}$ ,  $S_A = \{boy\}$ ,  $\leq$  is the transitive and reflexive of  $\{(boy, person), (boy, male), (boy, animal), (person, animal), (male, animal)\}$ ,  $C = \{bob\}$ ,  $P = \{p_{person}, p_{male}, p_{animal}, p_{boy}, p_{\top}\}$ ,  $\Omega = \{bob: \rightarrow person, p_{person}: animal, p_{male}: animal, p_{animal}: \top, p_{boy}: person, p_{\top}: \top\}$ , and  $boy$  is time dependent in  $\Sigma$ . The formula  $\neg F'$  is transformed into an equivalent one in negation normal form as follows:

$$\begin{aligned} \neg(\blacklozenge_{\mathbf{Tim}} p_{boy}(bob_{person}) \rightarrow \blacklozenge_{\mathbf{Tim}}(\exists x_{person})(\Box_{\mathbf{Sit}} p_{male}(x_{person}))) \\ \simeq \blacklozenge_{\mathbf{Tim}} p_{boy}(bob_{person}) \wedge \neg \blacklozenge_{\mathbf{Tim}}(\exists x_{person})(\Box_{\mathbf{Sit}} p_{male}(x_{person})) \\ \simeq \blacklozenge_{\mathbf{Tim}} p_{boy}(bob_{person}) \wedge \Box_{\mathbf{Tim}} \neg(\exists x_{person})(\Box_{\mathbf{Sit}} p_{male}(x_{person})) \\ \simeq \blacklozenge_{\mathbf{Tim}} p_{boy}(bob_{person}) \wedge \Box_{\mathbf{Tim}}(\forall x_{person})(\neg \Box_{\mathbf{Sit}} p_{male}(x_{person})) \\ \simeq \blacklozenge_{\mathbf{Tim}} p_{boy}(bob_{person}) \wedge \Box_{\mathbf{Tim}}(\forall x_{person}) \blacklozenge_{\mathbf{Sit}}(\neg p_{male}(x_{person})) \end{aligned}$$

In Figure 4, we show a proof of testing the satisfiability of the formula  $\neg F'$ . Since every tableau for  $S' = \{\neg F'\}$  is closed,  $F'$  is  $\Sigma^+$ -valid.

## 5 Completeness

This section proves the completeness of our proposed tableau calculus by combining the techniques in order-sorted logic [1, 19] and in quantified modal logic [4].

**Lemma 1** *Let  $M = (W, w_0, R, R', U, I)$  be a sorted  $\Sigma^+$ -structure and  $A$  be a closed formula. For every  $w \in W$ , the following statements hold:*

1.  $A \in Nex_w$  if and only if  $w \not\models \mathcal{E}(A)$  and  $\mathcal{E}(A) \neq \emptyset$ .
2. If  $A$  dose not contain any ground term, then  $A \notin Nex_w$  and  $\mathcal{E}(A) = \emptyset$ .

**Proof** By definition, it is trivial. ■

Let  $A$  be a closed formula. We define  $A \wedge \mathcal{E}(A) = A$  and  $A \vee \neg\mathcal{E}(A) = A$  if  $\mathcal{E}(A) = \emptyset$ . Let  $S$  be a set of closed formulas. We denote  $S \wedge \mathcal{E}(S)$  as  $\{F \wedge \mathcal{E}(F) \mid F \in S\}$  and  $S \vee \neg\mathcal{E}(S)$  as  $\{F \vee \neg\mathcal{E}(F) \mid F \in S\}$ .

**Lemma 2** Let  $M = (W, w_0, R, R', U, I)$  be a sorted  $\Sigma^+$ -structure and let  $w \in W$ . If  $\llbracket t \rrbracket_w = d$ , then  $w \models A[x_s/\bar{d}] \Leftrightarrow w \models A[x_s/t]$ .

**Proof** Similar to the proof of the counterpart in [4]. ■

The sorted  $\Sigma^+$ -structures result in the fact that every  $\Sigma^+$ -valid closed formula  $F$  implies the  $\Sigma^+$ -validity of the modal formula  $\blacksquare F$ , and vice versa if  $F$  dose not contain any ground term.

**Proposition 3** Let  $F$  be a closed formula. (i) if  $F$  is  $\Sigma^+$ -valid, then  $\blacksquare F$  is  $\Sigma^+$ -valid. (ii) if  $\blacksquare F$  is  $\Sigma^+$ -valid and contains no ground terms, then  $F$  is  $\Sigma^+$ -valid.

**Proof** (i) Let  $M = (W, w_0, R, R', U, I)$  be a sorted  $\Sigma^+$ -structure such that  $w_0 \not\models \blacksquare F$ . Then, for some  $w' \in W$  with  $(w_0, w') \in R'$ ,  $w' \not\models F$  and  $F \notin Nex_{w'}$ . Let us set the sorted  $\Sigma^+$ -structure  $M' = (W, w', R, R', U, I)$ . It does not satisfy  $F$  in  $w'$ , i.e.,  $w' \not\models F$ . (ii) Suppose that for every sorted  $\Sigma^+$ -structure  $M = (W, w_0, R, R', U, I)$ ,  $w_0 \models \blacksquare F$ . So, for all  $w' \in W$  with  $(w_0, w') \in R'$ ,  $w' \models F$  or  $F \in Nex_{w'}$ . If  $F$  does not contain any ground term, then  $F \notin Nex_w$  for every  $w \in W$  (by Lemma 1 (2)). Since  $R'$  is reflexive (i.e.,  $(w_0, w_0) \in R'$ ) and  $F \notin Nex_{w_0}$ , we have  $w_0 \models F$ . ■

The following theorem shows the soundness of the prefixed tableau calculus.

**Theorem 1** If there exists a closed tableau for  $S$ , then  $S$  is  $\Sigma^+$ -unsatisfiable.

**Proof** Suppose that  $S$  is  $\Sigma^+$ -satisfiable. Then, there exists a sorted  $\Sigma^+$ -structure  $M = (W, w_0, R, R', U, I)$  such that  $w_0 \models S$ . We will prove that there is no closed tableau for  $S$ . Let  $T$  be any tableau for  $S$ . It suffices to show that there exists a sub-tableau  $T'$  of  $T$  such that the root  $S_0 = (S^+)^1$ , each non-leaf node  $S_k$  has only one child  $S_{k+1}$ , and every node is  $\Sigma^+$ -satisfiable (which implies that every node is not closed because every closed node is  $\Sigma^+$ -unsatisfiable). We show the satisfiability of each node  $S_k$  by induction on the depth  $k$  of the tableau  $T$ .

Base case:  $k = 0$ . By Proposition 1 and Proposition 3, the formulas  $\blacksquare F$  for all axioms  $F$  are  $\Sigma^+$ -valid, and by the assumption,  $M$  satisfies all the formulas in  $S$ . So,  $w_0 \models S^+$  where  $S^+$  is the non-annotated set of  $(S^+)^1$ .

Induction step:  $k > 0$ .

( $\nu_j$ -rule) Let us assume  $w \models \{\Box_j A\} \cup S'$  where  $j \in \{Tim, Sit\}$  and  $w \in W_j$ . Then, for all  $w' \in W_j$  with  $(w, w') \in R_j$ ,  $w' \models A$  or  $A \in Nex_{w'}$ . Since  $R_j$  is reflexive over  $W_j \times W_j$ ,  $(w, w) \in R_j$ , and so  $w \models A$  or  $A \in Nex_w$ . This derives  $w \models A \vee \neg\mathcal{E}(A)$  by Lemma 1 (1).

( $\pi_j$ -rule) Let us assume  $w \models \{\diamond_j A\} \cup \square_j(S' \cup S'') \cup \blacksquare(S' \cup S'')$  where  $j \in \{Tim, Sit\}$  and  $w \in W_j$ . For some  $w_a \in W_j$  with  $(w, w_a) \in R_j$ ,  $w_a \models A$  and  $A \notin Nex_{w_a}$ , and thus  $w_a \models A \wedge \mathcal{E}(A)$ . Let  $F \in \{F \mid \square_j F \in \square_j S' \cup \blacksquare S'\}$ . By the assumption, for all  $w' \in W_j$  with  $(w, w') \in R_j$ ,  $w' \models F$  or  $F \in Nex_{w'}$ . So,  $w_a \models F \vee \neg \mathcal{E}(F)$  by Lemma 1 (1). Moreover, since  $R_j$  is transitive over  $W_j \times W_j$ , if  $(w, w_a)$  and  $(w_a, w'_a)$  in  $R_j$ , then  $(w, w'_a)$  in  $R_j$ . This implies that for all  $w'' \in W_j$  with  $(w_a, w'') \in R_j$ ,  $w'' \models F$  or  $F \in Nex_{w''}$  (i.e.,  $w_a \models \square_j F$ ). Let  $F \in \{F \mid \blacksquare F \in \square_j S' \cup \blacksquare S'\}$ . By the assumption, for all  $w' \in W$  with  $(w, w') \in R'$ ,  $w' \models F$  or  $F \in Nex_{w'}$ . Since  $(w, w_a) \in R_j (\subseteq R')$ ,  $w_a \models F \vee \neg \mathcal{E}(F)$  by Lemma 1 (1). Since  $R'$  is transitive, if  $(w, w_a)$  and  $(w_a, w'_a)$  in  $R'$ , then  $(w, w'_a)$  in  $R'$ . Hence, for all  $w'' \in W$  with  $(w_a, w'') \in R'$ ,  $w'' \models F$  or  $F \in Nex_{w''}$  (i.e.,  $w_a \models \blacksquare F$ ).

( $\pi_{i \rightarrow j}$ -rule) Let us assume  $w \models \{\diamond_j A\} \cup \square_j(S' \cup S'') \cup \blacksquare(S' \cup S'')$  with  $i \neq j$  where  $w \in W$  if  $i = W$ ,  $w \in W_i$  otherwise. For some  $w_a \in W_j$  with  $(w, w_a) \in R_j$ ,  $w_a \models A$  and  $A \notin Nex_{w_a}$ . So,  $w_a \models A \wedge \mathcal{E}(A)$ . Let  $F \in \{F \mid \square_j F \in \square_j S' \cup \blacksquare S'\}$ . By the assumption, for all  $w' \in W_j$  with  $(w, w') \in R_j$ ,  $w' \models F$  or  $F \in Nex_{w'}$ . So,  $w_a \models F \vee \neg \mathcal{E}(F)$  by Lemma 1 (1). Let  $F \in \{F \mid \blacksquare F \in \square_j S' \cup \blacksquare S'\}$ . By the assumption, for all  $w' \in W$  with  $(w, w') \in R'$ ,  $w' \models F$  or  $F \in Nex_{w'}$ . Because  $(w, w_a) \in R_j (\subseteq R')$ ,  $w_a \models F \vee \neg \mathcal{E}(F)$  by Lemma 1 (1). Moreover, since  $R'$  is transitive, if  $(w, w_a)$  and  $(w_a, w'_a)$  in  $R'$ , then  $(w, w'_a)$  in  $R'$ . This implies that for all  $w'' \in W$  with  $(w_a, w'') \in R'$ ,  $w'' \models F$  or  $F \in Nex_{w''}$  (i.e.,  $w_a \models \blacksquare F$ ).

( $\pi_W$ -rule) Let us assume  $w \models \{\blacklozenge A\} \cup \blacksquare S' \cup S''$ . For some  $w_a \in W$  with  $(w, w_a) \in R'$ ,  $w_a \models A$  and  $A \notin Nex_{w_a}$ . Hence,  $w_a \models A \wedge \mathcal{E}(A)$ . Let  $F \in \{F \mid \blacksquare F \in \blacksquare S'\}$ . By the assumption, for all  $w' \in W$  with  $(w, w') \in R'$ ,  $w' \models F$  or  $F \in Nex_{w'}$ . So,  $w_a \models F \vee \neg \mathcal{E}(F)$  by Lemma 1 (1). Moreover, since  $R'$  is transitive, if  $(w, w_a)$  and  $(w_a, w'_a)$  in  $R'$ , then  $(w, w'_a)$  in  $R'$ . Therefore, for all  $w'' \in W$  with  $(w_a, w'') \in R'$ ,  $w'' \models F$  or  $F \in Nex_{w''}$  (i.e.,  $w_a \models \blacksquare F$ ).

( $\blacksquare \square$ -rule) Let us assume  $w \models \{\blacksquare A\} \cup S'$ . For all  $w' \in W$  with  $(w, w') \in R'$ ,  $w' \models A$  or  $A \in Nex_{w'}$ .  $R_{Tim}$  and  $R_{Sit}$  are subsets of  $R'$ . So, for all  $w' \in W$  with  $(w, w') \in R_{Tim} \cup R_{Sit}$ ,  $w' \models A$  or  $A \in Nex_{w'}$ . Hence,  $w \models \square_{Tim} A$  and  $w \models \square_{Sit} A$ . Since  $R'$  is reflexive,  $(w, w) \in R'$ , and by Lemma 1 (1),  $w \models A \vee \neg \mathcal{E}(A)$ .

( $\diamond \blacklozenge$ -rule) Let us assume  $w \models \{\diamond_j A\} \cup S'$ . For some  $w_a \in W_j$  with  $(w, w_a) \in R_j$ ,  $w_a \models A$  and  $A \notin Nex_{w_a}$  (i.e.,  $w_a \models A \wedge \mathcal{E}(A)$ ). Due to  $R_j \subseteq R'$  and  $W_j \subseteq W$ ,  $(w, w_a) \in R'$  and  $w_a \in W$ . From this  $w \models \blacklozenge A$  follows.

( $\gamma_\tau$ -rule) Let  $w \models \{\forall x_\tau A\} \cup S'$ . For all  $d \in I_w(\tau)$ ,  $w \models A[x_\tau/\bar{d}]$ . Since  $sort(t) \leq \tau$  and  $t$  is a ground term with level  $n$ ,  $\llbracket t \rrbracket_w = d' \in U_w \cap I(\tau) (\subseteq I_w(\tau))$ . So,  $w \models A[x_\tau/\bar{d}']$ . This yields  $w \models A[x_\tau/t]$ .

( $\gamma_s$ -rule) Let  $w \models \{p_{s'}(t), \forall x_s A\} \cup S'$  where  $s' \leq s$ . Then,  $\llbracket t \rrbracket_w \in I_w(p_{s'})$ , and for all  $d \in I_w(s)$ ,  $w \models A[x_s/\bar{d}]$ . If  $s$  is extensible, then  $p_{s'}$  is an anti-rigid sort predicate. So,  $I_w(p_{s'}) = I_w(s') (\subseteq I_w(s))$ . If  $s$  is inextensible, then  $(I_w(p_{s'}) \subseteq) I_w(p_s) = I_w(s)$ . Thus,  $\llbracket t \rrbracket_w = d' \in I_w(s)$ , and  $w \models A[x_s/\bar{d}']$ . It follows that  $w \models A[x_s/t]$ .

( $\delta_\tau$ -rule) Let  $w \models \{\exists x_\tau A\} \cup S'$ . For some  $d \in I_w(\tau)$ ,  $w \models A[x_\tau/\bar{d}]$ . Let us extend  $M$  to  $M'$  by setting  $I_w(c) = d (\in I_w(\tau) \subseteq U_w)$  where  $c_\tau$  is not in  $\{\exists x_\tau A\} \cup S'$  and  $c: \rightarrow \tau \in \Omega$ . Therefore,  $w \models E(c_\tau)$  and  $w \models A[x_\tau/c_\tau]$  in  $M'$ .

( $\delta_\sigma$ -rule) Let  $w \models \{\exists x_\sigma A\} \cup S'$  where  $p_\sigma: \tau \in \Omega$ . For some  $d \in I_w(\sigma)$ ,  $w \models A[x_\sigma/\bar{d}]$ . Let us extend  $M$  to  $M'$  by setting  $I_w(c) = d (\in I_w(\sigma))$  where  $c_\tau$  is not in  $\{\exists x_\sigma A\} \cup S'$  and  $c: \rightarrow \tau \in \Omega$ . Thus,  $w \models A[x_\sigma/c_\tau]$  in  $M'$ . Because of  $I_w(\sigma) = I_w(p_\sigma)$ ,  $w \models p_\sigma(c_\tau)$  in  $M'$ .

( $p_\tau$ -rule) Let  $w \models S'$ , and let  $t$  be a ground term level  $n$  such that  $sort(t) \leq \tau$ . Then,  $\llbracket t \rrbracket_w \in U_w \cap I(\tau)$ , and by the definition of sorted  $\Sigma$ -structures,  $\llbracket t \rrbracket_w \in I_w(\tau)$ . Since  $I_w(\tau) \subseteq I_w(p_\tau)$ , we obtain  $w \models p_\tau(t)$ .

( $<$ -rule) Let  $w \models \{p_s(t)\} \cup S'$ . If  $s \leq s'$ , then by definition,  $I_w(p_s) \subseteq I_w(p_{s'})$ . Hence,  $w \models p_{s'}(t)$ .

( $E$ -rule) Let us assume  $w \models \{\neg E(t)\} \cup S'$  where  $t$  is a ground term with level  $n$ . Then,  $t^n$  occurs in a positive atomic formula (i.e.,  $S'$  contains (i) an atomic formula  $p(t_1^n, \dots, t_l^n)$  where  $t = t_u$  for some  $u \in \{1, \dots, l\}$  or (ii)  $E(t^n)$ ). For (i),  $(\llbracket t_1 \rrbracket_w, \dots, \llbracket t_l \rrbracket_w) \in I_w(p)$  ( $\subseteq U_w \times \dots \times U_w$ ), so that  $\llbracket t \rrbracket_w \in U_w$ . Thus, (i) and (ii) imply  $w \models E(t)$ . It is a contradiction.

( $\alpha$ -rule), ( $\beta$ -rule), and ( $I$ -rule) For the cases, it is easy to show that each tableau rule preserves satisfiability.

Therefore, since every node  $S_k$  in the sub-tableau  $T'$  is  $\Sigma^+$ -satisfiable, it is not closed.  $\blacksquare$

In order to prove the completeness of the prefixed tableau calculus, we need to define saturated sets of formulas and a canonical interpretation of a formula set. Let  $(i, n): S$  be a prefixed set of closed formulas. We denote the set of ground terms of sort  $s$  with level  $n$  in  $S$  by  $\mathcal{T}_{s,0}((i, n): S) = \{t \in \mathcal{T}_0 \cap \mathcal{T}_s \mid t^n \text{ occurs in } S\} \cup \{a_n\}$  where  $a_n$  is the dummy constant for level  $n$ . We represent the set of ground terms of a sort predicate  $p_s$  with level  $n$  in  $S$  by  $\mathcal{T}_{p_s,0}((i, n): S)$  such that (i) if  $p_s$  is inextensible then  $\mathcal{T}_{p_s,0}((i, n): S) = \bigcup_{s' \leq s} \{t \in \mathcal{T}_0 \mid p_{s'}(t^n) \in S\}$  and (ii) if  $p_s$  is extensible then  $\mathcal{T}_{p_s,0}((i, n): S) = \bigcup_{s' \leq s} \{t \in \mathcal{T}_0 \mid p_{s'}(t^n) \in S \text{ and } p_{s'} \text{ is an inextensible sort predicate}\}$ . The set  $\mathcal{T}_{p_s,0}((i, n): S)$  consists of the ground terms  $t$  in atomic sort predicate formulas  $p_{s'}(t^n)$  for all the sorts  $s'$  with  $s' \leq s^6$  (where the target sort predicates  $p_{s'}$  are inextensible when the sort predicate  $p_s$  is extensible, they are any sort predicates otherwise). For example, let  $p_\tau$  be an extensible type predicate and  $p_\sigma$  be an (inextensible) anti-rigid sort predicate where  $\sigma \leq \tau$ . Consider the case where  $p_\tau(t_1^n) \in S$  and  $p_\sigma(t_2^n) \in S$ . Then, the set  $\mathcal{T}_{p_\tau,0}((i, n): S)$  does not contain  $t_1$  but contains  $t_2$  since  $p_\tau$  is extensible and  $\sigma$  is an inextensible subsort of  $\tau$ .

**Lemma 3** *Let  $\Sigma = (T, S_A, \Omega, \leq)$  be a sorted signature and  $(i, n): S$  be a prefixed set of closed formulas. If  $s \leq s'$ , then  $\mathcal{T}_{s,0}((i, n): S) \subseteq \mathcal{T}_{s',0}((i, n): S)$  and  $\mathcal{T}_{p_s,0}((i, n): S) \subseteq \mathcal{T}_{p_{s'},0}((i, n): S)$ .*

**Proof** By definition, it is easy to show.  $\blacksquare$

A set  $S$  of closed formulas is consistent if its non-annotated set does not contain any pair of complementary literals (i.e.,  $\neg A$  and  $A$ ) or the clash symbol  $\perp$ . For example, if  $\{\neg p(t), p(t^n)\} \subseteq S$ , then it is not consistent.

**Definition 16** *A prefixed set  $(i, n): S$  is saturated if  $S$  is consistent and the following rules are satisfied:*

1.  $p_\tau(t^n) \in S$  for every ground term  $t \in \mathcal{T}_{\tau,0}((i, n): S)$ .
2. If  $p_s(t^n) \in S$  and  $s \leq s'$ , then  $p_{s'}(t^n) \in S$ .
3. If  $F_1 \wedge F_2 \in S$ , then  $F_1^n \in S$  and  $F_2^n \in S$ .
4. If  $F_1 \vee F_2 \in S$ , then either  $F_1^n \in S$  or  $F_2^n \in S$ .
5. If  $(\forall x_s)F \in S$ , then  $F[x_s/t]^n \in S$  for every ground term  $t \in \mathcal{T}_{s,0}((i, n): S) \cup \mathcal{T}_{p_s,0}((i, n): S)$ .

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<sup>6</sup>Every subsort relation is reflexive, antisymmetric, and transitive since it is a partial order.

6. If  $(\exists x_s)F \in S$ , then  $p_s(c_\tau^n), F[x_s/c_\tau]^n \in S$  for some constant  $c_\tau$  where  $\tau = s$  if  $s \in T$ ,  $p_s: \tau \in \Omega$  otherwise.
7. If  $\Box_i F \in S$  with  $i \in \{Tim, Sit\}$ , then  $F \vee \neg \mathcal{E}(F) \in S$ .
8. If  $\blacksquare F \in S$ , then  $F \vee \neg \mathcal{E}(F), \Box_{Tim} F, \Box_{Sit} F \in S$ .
9. If  $\Diamond_i F \in S$  with  $i \in \{Tim, Sit\}$ , then  $\blacklozenge F \in S$ .

A prefixed set  $(i, n): S$  is *tab-consistent* if for every finite subset  $S'$  of  $S$  there is no closed tableau for  $(i, n): S'$ .

**Lemma 4** *If  $(i, n): S$  is tab-consistent, then there exists a superset  $S^*$  of  $S$  such that  $(i, n): S^*$  is saturated.*

The following lemma will be used to construct a canonical interpretation.

**Lemma 5** *Let  $(i, n): S$  be tab-consistent and let  $j \in \{Tim, Sit\}$ . The following statements hold:*

1. if  $S = \Diamond_j A, \Box_j S' \cup \blacksquare S', S''$  ( $i = j$ ), then  $(j, n+1): A \wedge \mathcal{E}(A), S' \vee \neg \mathcal{E}(S'), \langle \Box_j, \blacksquare \rangle S'$  is tab-consistent.
2. if  $S = \Diamond_j A, \Box_j S' \cup \blacksquare S', S''$  ( $i \neq j$ ), then  $(j, n+1): A \wedge \mathcal{E}(A), S' \vee \neg \mathcal{E}(S'), \blacksquare S'$  is tab-consistent.
3. if  $S = \blacklozenge A, \blacksquare S', S''$ , then  $(W, n+1): A \wedge \mathcal{E}(A), S' \vee \neg \mathcal{E}(S'), \blacksquare S'$  is tab-consistent.

We now are ready to define a canonical interpretation of a formula set.

**Definition 17** *Let  $S$  be a finite set of closed formulas and let the initial set  $(W, 1): (S^+)^1$  be tab-consistent. The canonical interpretation of  $(S^+)^1$  is a tuple  $M_c = (W, w_0, R, R', U, I)$  fulfilling the following conditions:*

1.  $W$  is a family of prefixed sets of closed formulas where
  - (a)  $w_0 = (W, 1): ((S^+)^1)^*$  such that  $((S^+)^1)^*$  is a superset of  $(S^+)^1$  and  $w_0$  is saturated (it exists by Lemma 4). Then,  $w_0$  is tab-consistent since  $(W, 1): (S^+)^1$  is tab-consistent.
  - (b) if  $w = (i, n): S_k$  ( $n > 0$ ), then
    - i. for every  $\Diamond_j A$  ( $i = j$ ) in  $S_k$  ( $= \{\Diamond_j A\} \cup \Box_j(S' \cup S'') \cup \blacksquare(S' \cup S'')$ ),  $w' = (j, n+1): S'_k$  such that  $S'_k$  is a superset of  $\{A \wedge \mathcal{E}(A)\} \cup (S' \vee \neg \mathcal{E}(S')) \cup \Box_j S' \cup \blacksquare S'$  and  $w'$  is saturated. By Lemma 5,  $(j, n+1): \{A \wedge \mathcal{E}(A)\} \cup (S' \vee \neg \mathcal{E}(S')) \cup \Box_j S' \cup \blacksquare S'$  is tab-consistent. Hence, by Lemma 4,  $w'$  exists and is tab-consistent. Moreover,  $(w, w') \in \mathbf{R}_j$ .
    - ii. for every  $\Diamond_j A$  ( $i \neq j$ ) in  $S_k$  ( $= \{\Diamond_j A\} \cup \Box_j(S' \cup S'') \cup \blacksquare(S' \cup S'')$ ),  $w' = (j, n+1): S'_k$  such that  $S'_k$  is a superset of  $\{A \wedge \mathcal{E}(A)\} \cup (S' \vee \neg \mathcal{E}(S')) \cup \blacksquare S'$  and  $w'$  is saturated. By Lemma 5,  $(j, n+1): \{A \wedge \mathcal{E}(A)\} \cup (S' \vee \neg \mathcal{E}(S')) \cup \blacksquare S'$  is tab-consistent. Hence, by Lemma 4,  $w'$  exists and is tab-consistent. Moreover,  $(w, w') \in \mathbf{R}_j$ .

iii. for every  $\blacklozenge A$  in  $S_k$  ( $= \{\blacklozenge A\} \cup \blacksquare S' \cup S''$ ),  $w' = (W, n+1): S'_k$  such that  $S'_k$  is a superset of  $\{A \wedge \mathcal{E}(A)\} \cup (S' \vee \neg \mathcal{E}(S')) \cup \blacksquare S'$  and  $w'$  is saturated. By Lemma 5,  $(W, n+1): \{A \wedge \mathcal{E}(A)\} \cup (S' \vee \neg \mathcal{E}(S')) \cup \blacksquare S'$  is tab-consistent. Hence, by Lemma 4,  $w'$  exists and is tab-consistent. Moreover,  $(w, w') \in \mathbf{R}'$ .

2.  $W_{\mathbf{Tim}} = \{w \in W \mid w = (\text{Tim}, n): S_k\}$  and  $W_{\mathbf{Sit}} = \{w \in W \mid w = (\text{Sit}, n): S_k\}$ ,
3.  $R = (R_{\mathbf{Tim}}, R_{\mathbf{Sit}})$  where  $R_{\mathbf{Tim}} = \text{cl}(\{(w, w') \in W_{\mathbf{Tim}} \times W_{\mathbf{Tim}} \mid (w, w') \in \mathbf{R}_{\mathbf{Tim}}\}) \cup \mathbf{R}_{\mathbf{Tim}}$  and  $R_{\mathbf{Sit}} = \text{cl}(\{(w, w') \in W_{\mathbf{Sit}} \times W_{\mathbf{Sit}} \mid (w, w') \in \mathbf{R}_{\mathbf{Sit}}\}) \cup \mathbf{R}_{\mathbf{Sit}}$ <sup>7</sup>,
4.  $R' = \text{cl}(\mathbf{R}_{\mathbf{Tim}} \cup \mathbf{R}_{\mathbf{Sit}} \cup \mathbf{R}')$ .
5.  $U$  is a superset of  $\bigcup_{w \in W} U_w$  where  $U_w = \{t \in \mathcal{T}_0 \mid t^n \text{ occurs in } S_k\} \cup \{a_n\}$  with  $w = (i, n): S_k$ .
6.  $I$  is the set of interpretation functions  $I_w$  for all worlds  $w \in W$  such that
  - (a)  $I_w(s) = \mathcal{T}_{s,0}(w) \cup \mathcal{T}_{p_s,0}(w)$ . In addition,  $I(s) = (\mathcal{T}_0 \cap \mathcal{T}_s) \cup \bigcup_{w \in W} I_w(s)$ ,
  - (b)  $I_w(c) = c_\tau$  where  $c: \rightarrow \tau \in \Omega$ ,
  - (c)  $I_w(f)(t_1, \dots, t_l) = f_{\tau^*, \tau}(t_1, \dots, t_l)$  with  $\tau^* = \tau_1, \dots, \tau_l$  where  $f: \tau_1 \times \dots \times \tau_l \rightarrow \tau \in \Omega$ ,
  - (d)  $I_w(p) = \{(t_1, \dots, t_l) \mid p(t_1^n, \dots, t_l^n) \in S_k\}$  with  $w = (i, n): S_k$ .

In the canonical interpretation, if  $p_s$  is an extensible type predicate (i.e.,  $s = \tau$  and  $\tau$  is situation dependent), then by definition,  $\{t \in \mathcal{T}_0 \mid p_s(t^n) \in S_k\} \not\subseteq \mathcal{T}_{p_s,0}(w)$  where  $w = (i, n): S_k$ . In the case, we have  $I_w(s) \subsetneq I_w(p_s)$ . In the condition (6-a), the set  $\mathcal{T}_0 \cap \mathcal{T}_s$  of ground terms of sort  $s$  is used to define  $I(s)$ . Since the terms in  $\mathcal{T}_0 \cap \mathcal{T}_s$  are not limited to level  $n$ , some of them may not exist in any world, i.e.,  $t \in I(s)$  but  $t \notin I_w(s)$  ( $\subseteq U_w$ ) for any  $w \in W$ .

The following lemma shows that each canonical interpretation satisfies the semantic conditions of time/situation/time-situation dependencies in sorted  $\Sigma^+$ -structures.

**Lemma 6** *If  $(W, 1): (S^+)^1$  is tab-consistent, then the canonical interpretation  $M_c$  of  $(S^+)^1$  satisfies the conditions of time/situation/time-situation dependencies (in Definitions 3-5).*

Let  $S$  be a finite set of closed formulas such that the initial tableau  $(W, 1): (S^+)^1$  is tab-consistent. The following lemma indicates model existence for  $S$ .

**Lemma 7** *Let  $(W, 1): (S^+)^1$  be tab-consistent and let  $M_c = (W, w_0, R, R', U, I)$  be the canonical interpretation of  $(S^+)^1$ . (i)  $M_c$  is a sorted  $\Sigma^+$ -structure, and (ii) for all  $F \in ((S^+)^1)^*$  with  $w_0 = (W, 1): ((S^+)^1)^*$ ,  $w_0 \models F'$  where  $F'$  is the non-annotated formula of  $F$ .*

**Theorem 2** *If  $S$  is  $\Sigma^+$ -unsatisfiable, then there exists a closed tableau for  $S$ .*

<sup>7</sup>Let  $r$  be an ordered set.  $\text{cl}(r)$  denotes the reflexive and transitive closure of  $r$ . This is used to make an accessibility relation reflexive and transitive over possible worlds.

**Proof** We show that if there exists no closed tableau for  $S$ , then  $S$  is  $\Sigma^+$ -satisfiable. By the assumption, there is no subset  $S'$  of  $(S^+)^1$  such that  $(W, 1): S'$  has a closed tableau, i.e.,  $(W, 1): (S^+)^1$  is tab-consistent. So, by Definition 17,  $S^1 \subseteq (S^+)^1 \subseteq ((S^+)^1)^*$  where  $w_0 = (W, 1): ((S^+)^1)^*$ , and by Lemma 7,  $w_0 \models S$ . Thus, it is  $\Sigma^+$ -satisfiable. ■

This theorem leads to the completeness of the tableau calculus as follows:

**Theorem 3 (Completeness)** *There exists a closed tableau for  $S$  if and only if  $S$  is  $\Sigma^+$ -unsatisfiable.*

**Proof** By Theorem 1 and Theorem 2, it can be proved. ■

## 6 Conclusion and Future Work

The main results of this paper are two methodological contributions: (i) a refinement of the ontological property classification by means of the individual existence and time and situation dependencies and (ii) an integration of sort predicates and sorted terms (in order-sorted logic), modalities and varying domains (in quantified modal logic), and temporal operators (in first-order temporal logic) in order to model the ontological distinctions among properties.

We formalized the syntax, semantics, and inference system (with ontological axioms) for an order-sorted modal logic as a well-suited formalism to deal with the ontological property classification. The formal semantics of properties is practically and theoretically useful in deciding the *ontological* and philosophical suitability of property descriptions in information systems and for guaranteeing *logical* consistency in reasoning about properties. For the ontological reasoning, we presented a prefixed tableau calculus by extending Cialdea-Mayer and Cerrito's calculus. New technical ideas had to be introduced in order to deal with the existential predicate, sorted expressions, and the ordered multi-modalities (i.e., temporal and situational modalities are subsorts of general modalities, as shown in Figure 2). The multi-modal operators with individual existence (ontologically interpreted in this paper) derive existential predicate formulas and process formulas prefixed by a pair  $(i, n)$  of the type of worlds  $i$  and a natural number  $n$  (as a world). Based on these ideas, our modal operator rules are complicatedly designed compared with standard modal tableau calculi. Moreover, the distinctions of types, anti-rigid sorts, and sort predicates and the type and sort hierarchies led us to design new inference rules such as sorted quantifier, sort predicate, and existential predicate rules.

Future work in this area concerns three extensions. The first is to combine the temporal operators 'always in the future,' 'always in the past,' 'until,' and 'since' with the individual existence. We expect that the combination more precisely characterizes the temporal features of properties in the real world. For example, defining the feature that the property *person* holds from the time of birth of each person until the time of death of that person is possible, but our work does not yet cover it.

The second is to consider reasoning on the higher-order expressions of properties by extending the order-sorted modal logic. This is useful when the same descriptions may play different roles, such as object, class, and attribute in ontology description languages (related to Hayes and Menzel's approach [17]). This offers the potential of higher-order reasoning

on taxonomic knowledge. For example, the property *human* can be regarded as a class of humans, and it simultaneously belongs to the species of animals as an instance of the property *animal species*.

Finally, we plan to provide an effective strategy for applying rules in our proposed tableau calculus in order to better implement the calculus. The ordinal tableau calculus is not efficient due to the simple reasoning based on the semantics. Consider finding inconsistent pairs of atomic formulas. When the sorted quantifier rule ( $\gamma_\tau$ -rule) is applied to a variable in a predicate formula  $p(x_\tau)$ , a term  $t$  that occurs in its negative predicate  $\neg p(t)$  should be primarily substituted for the variable  $x_\tau$ . This strategy is similar to the reasoning strategy of the resolution system in logic programming used as a computer language. Hence, many efficient tableau calculi are optimized by incorporating such resolution steps and other techniques.

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## Appendix

**Proof of Proposition 1** Let  $M = (W, w_0, R, R', U, I)$  be any sorted  $\Sigma^+$ -structure. We prove that it satisfies the axioms.

(rigid predicate axiom) Let  $d$  be any individual in  $I_{w_0}(\tau)$  ( $\subseteq U_{w_0}$ ). If  $p_{\tau'}$  is an inextensible type predicate with  $p_{\tau'}: \tau \in \Omega$  and  $w_0 \models p_{\tau'}(\bar{d})$ , then  $d \in I_{w_0}(\tau')$  (due to the condition  $I_w(\tau') = I_w(p_{\tau'})$  for every  $w \in W$ ). So, by Definition 13 (2), for all  $w'$  with  $(w_0, w') \in R'$ ,  $d \in U_{w'}$  implies  $d \in I_{w'}(\tau')$  ( $= I_{w'}(p_{\tau'})$ ). So,  $p_{\tau'}(\bar{d}) \in Nex_{w'}$  (i.e.,  $\llbracket \bar{d} \rrbracket_{w'} \notin U_{w'}$ ) or  $w' \models p_{\tau'}(\bar{d})$ . Hence,  $w_0 \models \blacksquare p_{\tau'}(\bar{d})$ . Also, if  $p$  is a generic predicate with  $p: undef \in \Omega$  and  $w_0 \models p(\bar{d})$ , then by Definition 13 (3),  $w_0 \models \blacksquare p(\bar{d})$ .

(anti-rigid predicate axiom) Let  $d \in I_{w_0}(\tau)$  ( $\subseteq U_{w_0}$ ). By Definition 13 (2), if  $d \in I_{w_0}(\sigma)$ , then there exists  $w_j \in W$  with  $(w_0, w_j) \in R'$  such that  $d \notin I_{w_j}(\sigma)$  with  $d \in U_{w_j}$ . Thus, for every  $d' \in I_{w_0}(\tau)$ ,  $w_0 \models p_\sigma(\bar{d}')$  implies that for some  $w_j \in W$  with  $(w_0, w_j) \in R'$ ,  $w_j \models \neg p_\sigma(\bar{d}')$  and  $\neg p_\sigma(\bar{d}') \notin Nex_{w_j}$  according to  $I_w(\sigma) = I_w(p_\sigma)$  for every  $w \in W$ . Therefore, for every  $d' \in I_{w_0}(\tau)$ ,  $w_0 \models p_\sigma(\bar{d}') \rightarrow \blacklozenge(\neg p_\sigma(\bar{d}'))$ .

(time dependency axiom) Let  $tm$  be any time in  $W_{\mathbf{Tim}}$  with  $(w_0, tm) \in R_{\mathbf{Tim}}$ , and let  $d \in I_{tm}(\tau)$  ( $\subseteq U_{tm}$ ). By the temporal unstability, for every time-dependent predicate  $p_\sigma$  with  $p_\sigma: \tau \in \Omega$ , if  $d \in I_{tm}(p_\sigma)$ , then there exists  $tm_j \in W_{\mathbf{Tim}}$  with  $(tm, tm_j) \in R_{\mathbf{Tim}}$  such that  $d \notin I_{tm_j}(p_\sigma)$  with  $d \in U_{tm_j}$ . So, for every  $d' \in I_{tm}(\tau)$ , if  $tm \models p_\sigma(\bar{d}')$ , then  $tm \models \blacklozenge_{\mathbf{Tim}}(\neg p_\sigma(\bar{d}'))$ . Thus,  $tm \models (\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \blacklozenge_{\mathbf{Tim}}(\neg p_\sigma(x_\tau)))$ . Therefore,  $w_0 \models \square_{\mathbf{Tim}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \blacklozenge_{\mathbf{Tim}}(\neg p_\sigma(x_\tau)))$ . Moreover, by the situational stability, if  $d \in I_{tm}(p_\sigma)$  with  $(w_0, tm) \in R_{\mathbf{Tim}}$ , then for all situations  $st \in W_{\mathbf{Sit}}$  with  $(tm, st) \in R_{\mathbf{Sit}}$ ,  $d \in U_{st}$  (iff  $p_\sigma(\bar{d}) \notin Nex_{st}$ ) implies  $d \in I_{st}(p_\sigma)$ . Then, for every  $d' \in I_{tm}(\tau)$ ,  $tm \models p_\sigma(\bar{d}') \rightarrow \square_{\mathbf{Sit}} p_\sigma(\bar{d}')$ , so that  $w_0 \models \square_{\mathbf{Tim}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \square_{\mathbf{Sit}} p_\sigma(x_\tau))$ .

(situation dependency axiom) Similar to the above proof of the time dependency axiom.

(time-situation dependency axiom) Let  $p_\sigma$  be a time-situation dependent predicate with  $p_\sigma: \tau \in \Omega$ . The first formula is  $\Sigma^+$ -valid because it is the same as the situation dependency axiom. Let  $st$  be any situation in  $W_{\mathbf{Sit}}$  with  $(w_0, st) \in R_{\mathbf{Sit}}$ , and let  $d \in I_{st}(\tau)$  ( $\subseteq U_{st}$ ). If  $st \models p_\sigma(\bar{d})$ , then by the situational unstability under time, for some  $tm_i, tm_j \in W_{\mathbf{Tim}}$  with  $(st, tm_i), (st, tm_j) \in R_{\mathbf{Tim}}$ ,  $d \in I_{tm_i}(p_\sigma)$  and  $d \notin I_{tm_j}(p_\sigma)$  with  $d \in U_{tm_i} \cap U_{tm_j}$ . Thus, for

every  $d' \in I_{st}(\tau)$ ,  $st \models p_\sigma(\bar{d})$  implies  $tm_i \models p_\sigma(\bar{d}')$  and  $tm_j \models \neg p_\sigma(\bar{d}')$  where  $p_\sigma(\bar{d}') \notin Nex_{tm_i}$  and  $\neg p_\sigma(\bar{d}') \notin Nex_{tm_j}$ . So,  $st \models (\forall x_\tau)(p_\sigma(x_\tau) \rightarrow (\diamond_{\mathbf{Tim}} p_\sigma(x_\tau) \wedge \diamond_{\mathbf{Tim}}(\neg p_\sigma(x_\tau))))$ . This derives the conclusion that  $w_0 \models \Box_{\mathbf{Sit}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow (\diamond_{\mathbf{Tim}} p_\sigma(x_\tau) \wedge \diamond_{\mathbf{Tim}}(\neg p_\sigma(x_\tau))))$ .

(part-of relation axiom) Let  $d_1, d_2 \in I_{w_0}(\top)$  ( $= U_{w_0}$ ) and let  $q$  be an anti-sortal predicate. By Definition 6, if  $d_1 \in I_{w_0}(q)$  with  $(d_2, d_1) \in I_{w_0}(\prec)$ , then  $d_2 \in I_{w_0}(p)$ . It follows that  $w_0 \models (\forall x)(\forall y)(q(x) \wedge (y \prec x) \rightarrow q(y))$ .  $\blacksquare$

**Proof of Lemma 6** Let the initial set  $(W, 1): (S^+)^1$  of  $S$  be tab-consistent. By Lemma 4, there is a superset  $((S^+)^1)^*$  of  $(S^+)^1$  such that  $(W, 1): ((S^+)^1)^*$  is saturated. Let the canonical interpretation  $M_c = (W, w_0, R, R', U, I)$  be given by  $w_0 = (W, 1): ((S^+)^1)^*$  in Definition 17. We have to show that  $M_c$  satisfies Definitions 3-5. The initial set  $(S^+)^1$  contains the following formulas:

$$\begin{aligned} & \blacksquare \Box_{\mathbf{Tim}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \diamond_{\mathbf{Tim}}(\neg p_\sigma(x_\tau))) \\ & \blacksquare \Box_{\mathbf{Tim}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \Box_{\mathbf{Sit}} p_\sigma(x_\tau)) \end{aligned}$$

Let  $w$  be any world in  $W$ . By Definition 17, every world is accessible from  $w_0$ , and so  $(w_0, w) \in R'$ .

(temporally unstable) For the first formula, by Definition 17 (1-b) and Lemma 1 (2),  $w = (i, n): S_k$  must contain  $\Box_{\mathbf{Tim}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \diamond_{\mathbf{Tim}}(\neg p_\sigma(x_\tau)))$ . Let  $tm$  be any time with  $(w, tm) \in R_{\mathbf{Tim}}$  and let  $t \in U_{tm}$ . If  $t \notin I_{tm}(\tau)$ , then by definition,  $t \notin I_{tm}(p_\sigma)$ . For the other case,  $tm = (Tim, n+m): S'_k$  involves  $(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \diamond_{\mathbf{Tim}}(\neg p_\sigma(x_\tau)))$ . By Definition 16 (5),  $p_\sigma(t') \rightarrow \diamond_{\mathbf{Tim}}(\neg p_\sigma(t')) \in S'_k$  for every ground term  $t' \in \mathcal{T}_{\tau,0}(tm) \cup \mathcal{T}_{p_\tau,0}(tm)(= I_{tm}(\tau))$ . By Definition 16 (4),  $\neg p_\sigma(t) \in S'_k$  or  $\diamond_{\mathbf{Tim}}(\neg p_\sigma(t)) \in S'_k$ . If  $t \in I_{tm}(p_\sigma)$ , then  $p_\sigma(t^{n+m}) \in S'_k$ . Since  $S'_k$  is consistent,  $\diamond_{\mathbf{Tim}}(\neg p_\sigma(t)) \in S'_k$ . By Definition 17 (1-b),  $\neg p_\sigma(t), E(t^{n+m+1}) \in S''_k$  and  $(tm, tm_j) \in R_{\mathbf{Tim}}$  where  $tm_j = (Tim, n+m+1): S''_k$ . This leads to  $t \notin I_{tm_j}(p_\sigma)$  with  $t \in U_{tm_j}$ .

(situationally stable over time) For the second formula, by Definition 17 (1-b) and Lemma 1 (2),  $w = (i, n): S_k$  contains  $\Box_{\mathbf{Tim}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \Box_{\mathbf{Sit}} p_\sigma(x_\tau))$ . Let  $tm$  be any time with  $(w, tm) \in R_{\mathbf{Tim}}$  and let  $t \in U_{tm}$ . If  $t \notin I_{tm}(\tau)$ , then by definition,  $t \notin I_{tm}(p_\sigma)$ . For the other case,  $tm = (Tim, n+m): S'_k$  includes  $(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \Box_{\mathbf{Sit}} p_\sigma(x_\tau))$ . By Definition 16 (5),  $p_\sigma(t') \rightarrow \Box_{\mathbf{Sit}} p_\sigma(t') \in S'_k$  for every ground term  $t' \in \mathcal{T}_{\tau,0}(tm) \cup \mathcal{T}_{p_\tau,0}(tm)(= I_{tm}(\tau))$ . By Definition 16 (4),  $\neg p_\sigma(t) \in S'_k$  or  $\Box_{\mathbf{Sit}} p_\sigma(t) \in S'_k$ . If  $t \in I_{tm}(p_\sigma)$ , then  $p_\sigma(t^{n+m}) \in S'_k$ . Since  $S'_k$  is consistent,  $\Box_{\mathbf{Sit}} p_\sigma(t) \in S'_k$ . By Definition 17 (1-b), for all  $st$  with  $(tm, st) \in R_{\mathbf{Sit}}$ ,  $st = (Sit, n+m+l): S''_k$  contains  $p_\sigma(t) \vee \neg \mathcal{E}(p_\sigma(t))$ , and thus  $p_\sigma(t^{n+m+l}) \in S''_k$  or  $\neg E(t) \in S''_k$  (i.e.,  $t \notin U_{st}$ ). Hence,  $t \in U_{st}$  implies  $t \in I_{st}(p_\sigma)$ .

Let  $(S^+)^1$  contain the following formulas:

$$\begin{aligned} & \blacksquare \Box_{\mathbf{Sit}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \diamond_{\mathbf{Sit}}(\neg p_\sigma(x_\tau))) \\ & \blacksquare \Box_{\mathbf{Sit}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \Box_{\mathbf{Tim}} p_\sigma(x_\tau)) \end{aligned}$$

Similar to the above, the conditions of situation dependency are satisfied.

Let  $(S^+)^1$  contain the following formulas:

$$\begin{aligned} & \blacksquare \Box_{\mathbf{Sit}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \diamond_{\mathbf{Sit}}(\neg p_\sigma(x_\tau))) \\ & \blacksquare \Box_{\mathbf{Sit}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow (\diamond_{\mathbf{Tim}} p_\sigma(x_\tau) \wedge \diamond_{\mathbf{Tim}}(\neg p_\sigma(x_\tau)))) \end{aligned}$$

Let  $w$  be any world in  $W$ . By Definition 17, every world is accessible from  $w_0$ , and so  $(w_0, w) \in R'$ .

(situationally unstable) the same as above.

(temporally stable under situation) According to Definition 17 (1-b) and Lemma 1 (2), if  $w = (i, n) : S_k$ , then  $S_k$  must contain  $\Box_{\text{Sit}}(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow (\diamond_{\text{Tim}}p_\sigma(x_\tau) \wedge \diamond_{\text{Tim}}(\neg p_\sigma(x_\tau))))$ . Let  $st$  be any situation with  $(w, st) \in R_{\text{Sit}}$  and let  $t \in U_{st}$ . If  $t \notin I_{st}(\tau)$ , then by Definition 17,  $t \notin I_{st}(p_\sigma)$ . For the other case,  $st = (Sit, n + m) : S'_k$  must contain  $(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow (\diamond_{\text{Tim}}p_\sigma(x_\tau) \wedge \diamond_{\text{Tim}}(\neg p_\sigma(x_\tau))))$ . By Definition 16 (5),  $p_\sigma(t') \rightarrow (\diamond_{\text{Tim}}p_\sigma(t') \wedge \diamond_{\text{Tim}}(\neg p_\sigma(t')) \in S'_k$  for every ground term  $t' \in \mathcal{T}_{\tau,0}(st) \cup \mathcal{T}_{p_\tau,0}(st) (= I_{st}(\tau))$ . Since  $st$  is saturated, by Definition 16 (4),  $\neg p_\sigma(t) \in S'_k$  or  $\diamond_{\text{Tim}}p_\sigma(t) \wedge \diamond_{\text{Tim}}(\neg p_\sigma(t)) \in S'_k$ . By Definition 17 (6-d), if  $t \in I_{st}(p_\sigma)$ , then  $p_\sigma(t^{n+m}) \in S'_k$ . Since  $S'_k$  is consistent,  $\diamond_{\text{Tim}}p_\sigma(t) \wedge \diamond_{\text{Tim}}(\neg p_\sigma(t)) \in S'_k$ , and thus  $\diamond_{\text{Tim}}p_\sigma(t) \in S'_k$  and  $\diamond_{\text{Tim}}(\neg p_\sigma(t)) \in S'_k$ . By Definition 17 (1-b), there exists  $(st, tm_i) \in R_{\text{Sit}}$  where  $tm_i = (Tim, n + m + 1) : S''_k$  and  $p_\sigma(t^{n+m+1}), E(t^{n+m+1}) \in S''_k$ , and there exists  $(st, tm_j) \in R_{\text{Sit}}$  where  $tm_j = (Tim, n + m + 1) : S'''_k$  and  $\neg p_\sigma(t), E(t^{n+m+1}) \in S'''_k$ . Therefore,  $t \in I_{tm_i}(p_\sigma)$  and  $t \notin I_{tm_j}(p_\sigma)$  with  $t \in U_{tm_i} \cap U_{tm_j}$ . ■

Note that in the proof of (temporally stable under situation)  $tm_i$  and  $tm_j$  have the same level  $n + m + 1$  but they are different nodes in the tree of possible worlds in the canonical interpretation  $M_c$ .

**Proof of Lemma 4** Let a prefixed set  $(i, n) : S$  be tab-consistent. We inductively construct a sequence  $(i, n) : S_0, (i, n) : S_1, \dots, (i, n) : S_k, \dots$  of nodes by applications of static tableau rules where  $S_0 = S$ .

$$\frac{(i, n) : S_k}{(i, n) : S_{k+1}}$$

( $k > 0$ ) Let a static tableau rule of the form  $\frac{(i, n) : S_k}{(i, n) : S_{k+1}}$  be applied. Since  $(i, n) : S_k$  is tab-consistent,  $(i, n) : S_{k+1}$  is tab-consistent. This is because every static rule does not delete any pair of complementary literals (i.e.,  $\neg A$  and  $A$ ) or the clash symbol  $\perp$ . Let a static

$$\frac{(i, n) : S_k}{(i, n) : S'}$$

tableau rule of the form  $\frac{(i, n) : S' \quad (i, n) : S''}{(i, n) : S_{k+1}}$  be applied. Since  $(i, n) : S_k$  is tab-consistent,  $(i, n) : S'$  or  $(i, n) : S''$  is tab-consistent. We define  $S_{k+1} = S'$  if  $(i, n) : S'$  is tab-consistent,  $S_{k+1} = S''$  otherwise.

It remains to prove that there exists a superset  $S^*$  of  $S$  such that  $(i, n) : S^*$  is saturated.

Let  $S^* = \bigcup_{k \in \mathbb{N}} S_k$ . Assume that the non-annotated set of  $S^*$  contains a pair of an atomic formula  $A$  and its negation  $\neg A$  or the clash  $\perp$ . For each set  $S_k$ , let us denote the set of literals and  $\perp$  occurring in  $S_k$  by  $L(S_k)$ . Since every static rule does not delete them,  $L(S_0) \subseteq L(S_1) \subseteq \dots \subseteq L(S_k) \subseteq \dots$ . By the assumption, there exists  $m \in \mathbb{N}$  such that  $S_m$  is not consistent. This is contradictory to the fact that every node  $(i, n) : S_k$  is tab-consistent. It follows that  $S^*$  is consistent.

We need to check that  $S^*$  satisfies the conditions (1)-(9) in Definition 16. (1) by  $p_\tau$ -rule, this condition is satisfied. (2) by  $<$ -rule, this condition is satisfied. (3) and (4) by  $\alpha$ -rule and  $\beta$ -rule, these conditions are satisfied. (5) let  $s = \tau$ . By  $\gamma_\tau$ -rule, for every  $t \in \mathcal{T}_{\tau,0}((i, n) : S^*)$ ,  $F[x_\tau/t]^n \in S^*$ . Let  $t \in \mathcal{T}_{p_\tau,0}((i, n) : S^*)$ . If  $\tau$  is extensible, then  $p_s(t^n)$  exists in  $S^*$  such that  $p_s$  is inextensible. The predicate  $p_s$  is an anti-rigid sort predicate because  $s$  is a subsort of  $\tau$  where every subsort of situation dependent sorts must be a situation dependent type (as extensible) or situation/time-situation dependent anti-rigid sort (as inextensible). Thus, by  $\gamma_s$ -rule,  $F[x_\tau/t]^n \in S^*$ . If  $\tau$  is inextensible, then obviously  $F[x_\tau/t]^n \in S^*$  by  $\gamma_s$ -rule. Let  $s = \sigma$ .  $\mathcal{T}_{\sigma,0}((i, n) : S^*)$  includes only the dummy constant  $a_n$  since  $\mathcal{T}_0 \cap \mathcal{T}_\sigma = \emptyset$  (by Definition 8). Thus, by  $I$ -rule,  $p_\sigma(a_n^n) \in S^*$  with  $\text{sort}(a_n) \leq \sigma$ . For every  $t \in \mathcal{T}_{p_\sigma,0}((i, n) : S^*)$ , there

exists  $\sigma'$  such that  $\sigma' \leq \sigma$  and  $p_{\sigma'}(t^n) \in S^*$ . Since  $p_\sigma$  and  $p_{\sigma'}$  are inextensible, by  $\gamma_s$ -rule,  $F[x_\sigma/t]^n \in S^*$ . (6) let  $s = \tau$ . By  $\delta_\tau$ -rule,  $E(c_\tau^n), F[x_\tau/c_\tau]^n \in S^*$ , and thus, by  $p_\tau$ -rule,  $p_\tau(c_\tau^n) \in S^*$ . Let  $s = \sigma$  with  $p_\sigma: \tau \in \Omega$ . By  $\delta_\sigma$ -rule,  $p_\sigma(c_\tau^n), F[x_\sigma/c_\tau]^n \in S^*$ . (7)-(9) by  $\nu_j$ -rule,  $\blacksquare\Box$ -rule, and  $\blacklozenge$ -rule, these conditions are satisfied.

These yield the conclusion that  $(i, n): S^*$  is saturated.  $\blacksquare$

### Proof of Lemma 5

(1) Let us suppose that  $(j, n+1): A \wedge \mathcal{E}(A), S' \vee \neg\mathcal{E}(S'), \Box_j S' \cup \blacksquare S'$  is not tab-consistent. Then, for some finite subset  $S_l$  of it, there exists a closed tableau for  $(j, n+1): S_l$ . We can set  $S_l = (S_1 \wedge \mathcal{E}(S_1)) \cup (S_2 \vee \neg\mathcal{E}(S_2)) \cup S_3 \cup S_4$  where  $S_1 \subseteq \{A\}$ ,  $S_2 \subseteq S'$ ,  $S_3 \subseteq \Box_j S'$ , and  $S_4 \subseteq \blacksquare S'$ . Let  $S'_l = \{A \wedge \mathcal{E}(A)\} \cup (S'_2 \vee \neg\mathcal{E}(S'_2)) \cup S'_3 \cup S'_4$  where  $S'_2 = S_2 \cup \{B \mid \Box_j B \in S_3\} \cup \{B \mid \blacksquare B \in S_4\}$ ,  $S'_3 = S_3 \cup \Box_j S_2$ , and  $S'_4 = S_4 \cup \blacksquare S_2$ . Because of  $S_l \subseteq S'_l$ , there exists a closed tableau for  $(j, n+1): S'_l$ . Due to  $\Box_j S'_2 \cup \blacksquare S'_2 = S'_3 \cup S'_4$ ,  $\pi_j$ -rule derives  $(j, n+1): S'_l$  from  $(i, n): \blacklozenge_j A, \Box_j S'_2 \cup \blacksquare S'_2$ . Since  $S'_2 \subseteq S'$ ,  $\{\blacklozenge_j A\} \cup \Box_j S'_2 \cup \blacksquare S'_2$  is a finite subset of  $S$ . Hence,  $S$  is not tab-consistent.

(2) Suppose that  $(j, n+1): A \wedge \mathcal{E}(A), S' \vee \neg\mathcal{E}(S'), \blacksquare S'$  is not tab-consistent. Then, for some finite subset  $S_l$  of it, there exists a closed tableau for  $(j, n+1): S_l$ . We can set  $S_l = (S_1 \wedge \mathcal{E}(S_1)) \cup (S_2 \vee \neg\mathcal{E}(S_2)) \cup S_3$  where  $S_1 \subseteq \{A\}$ ,  $S_2 \subseteq S'$ , and  $S_3 \subseteq \blacksquare S'$ . Let  $S'_l = \{A \wedge \mathcal{E}(A)\} \cup (S'_2 \vee \neg\mathcal{E}(S'_2)) \cup S'_3$  where  $S'_2 = S_2 \cup \{B \mid \blacksquare B \in S_3\}$  and  $S'_3 = S_3 \cup \blacksquare S_2$ . Because of  $S_l \subseteq S'_l$ , there exists a closed tableau for  $(j, n+1): S'_l$ . Due to  $\blacksquare S'_2 = S'_3$ ,  $\pi_{i \rightarrow j}$ -rule infers  $(j, n+1): S'_l$  from  $(i, n): \blacklozenge_j A, \Box_j S'_2 \cup \blacksquare S'_2$ . Since  $S'_2 \subseteq S'$ ,  $\{\blacklozenge_j A\} \cup \Box_j S'_2 \cup \blacksquare S'_2$  is a finite subset of  $S$ . Thus,  $S$  is not tab-consistent.

(3) Let us suppose that  $(W, n+1): A \wedge \mathcal{E}(A), S' \vee \neg\mathcal{E}(S'), \blacksquare S'$  is not tab-consistent. Then, for some finite subset  $S_l$  of it, there exists a closed tableau for  $(W, n+1): S_l$ . We can set  $S_l = (S_1 \wedge \mathcal{E}(S_1)) \cup (S_2 \vee \neg\mathcal{E}(S_2)) \cup S_3$  where  $S_1 \subseteq \{A\}$ ,  $S_2 \subseteq S'$ , and  $S_3 \subseteq \blacksquare S'$ . Moreover, let  $S'_l = \{A \wedge \mathcal{E}(A)\} \cup (S'_2 \vee \neg\mathcal{E}(S'_2)) \cup S'_3$  where  $S'_2 = S_2 \cup \{B \mid \blacksquare B \in S_3\}$  and  $S'_3 = S_3 \cup \blacksquare S_2$ . Because of  $S_l \subseteq S'_l$ , there exists a closed tableau for  $(W, n+1): S'_l$ . Due to  $\blacksquare S'_2 = S'_3$ ,  $\pi_W$ -rule derives  $(W, n+1): S'_l$  from  $(i, n): \blacklozenge A, \blacksquare S'_2$ . Since  $S'_2 \subseteq S'$ ,  $\{\blacklozenge A\} \cup \blacksquare S'_2$  is a finite subset of  $S$ . It follows that  $S$  is not tab-consistent.  $\blacksquare$

**Proof of Lemma 7** Let  $M_c = (W, w_0, R, R', U, I)$  be the canonical interpretation of  $(S^+)^1$ .

(i) We show that  $M_c$  satisfies the conditions in the definition of sorted  $\Sigma$ -structures and in Definition 13 (i.e., sorted  $\Sigma^+$ -structures).

(sorted  $\Sigma$ -structure) The dummy constants make the domain of each world non-empty. So, by Definition 17 (1)-(2), condition (i) is satisfied. By Definition 17 (1)-(3), condition (ii) is satisfied, and by Definition 17 (4), condition (iii) is satisfied. By Definition 17 (5), condition (iv) is satisfied.

(v-1) Let  $s \in T \cup S_A$ . By Definition 17 (5) and (6-a),  $I_w(s) = \mathcal{T}_{s,0}(w) \cup \mathcal{T}_{p_s,0}(w) \subseteq U_w$ . In addition, by Definition 17 (6-a),  $I(s)$  is a superset of  $\bigcup_{w \in W} I_w(s)$  where  $U_w \cap I(s) \subseteq I_w(s)$ .

(v-2) Let  $t \in I_w(s)$  with  $s \leq s'$ . By Definition 17 (6-a),  $t \in \mathcal{T}_{s,0}(w) \cup \mathcal{T}_{p_s,0}(w) (\subseteq U_w)$ . By Lemma 3, if  $t \in \mathcal{T}_{s,0}(w)$  then  $t \in \mathcal{T}_{s',0}(w)$ , and if  $t \in \mathcal{T}_{p_s,0}(w)$  then  $t \in \mathcal{T}_{p_{s'},0}(w)$ . Thus, we have  $t \in I_w(s')$ .

(v-3) By Definition 17 (6-a) and (6-b), for every  $c \in C$  with  $c: \rightarrow \tau \in \Omega$ ,  $I_w(c) = c_\tau \in \mathcal{T}_0 \cap \mathcal{T}_\tau \subseteq I(\tau)$ .

(v-4) By Definition 17 (6-a) and (6-c), for every  $f \in F_n$  with  $f: \tau_1 \times \dots \times \tau_l \rightarrow \tau \in \Omega$ ,  $I_w(f)(t_1, \dots, t_l) = f_{\tau^*, \tau}(t_1, \dots, t_l) \in \mathcal{T}_0 \cap \mathcal{T}_\tau \subseteq I(\tau)$  where  $\tau^* = \tau_1, \dots, \tau_n$ .

(v-5) and (v-6) Let  $(t_1, \dots, t_l) \in I_w(p)$  where  $w = (i, n): S_k$  and  $p: s_1 \times \dots \times s_l \in \Omega$ . By Definition 17 (6-d),  $p(t_1^n, \dots, t_l^n) \in S_k$  where  $t_u \in \mathcal{T}_0 \cap \mathcal{T}_{s_u}$  for all  $u \in \{1, \dots, l\}$ . By condition (v-1),  $U_w \cap I(s) \subseteq I_w(s)$  for every  $s \in T \cup S_A$ . Hence,  $I_w(p) \subseteq I_w(s_1) \times \dots \times I_w(s_l)$ . Let  $t \in I_w(p_s)$  where  $w = (i, n): S_k$  and  $p_s: \tau \in \Omega$ . By Definition 17 (6-d),  $p_s(t^n) \in S_k$  where  $t \in \mathcal{T}_0 \cap \mathcal{T}_\tau$ . So, by (v-1),  $t \in I_w(\tau)$ , and hence  $I_w(p_s) \subseteq I_w(\tau)$  by Definition 17 (6-a). Let  $t \in I_w(p)$  where  $p \in P_{non}$  and  $p: undef \in \Omega$ . By Definition 17 (6-d),  $t \in U_w \cup \Delta_w$ .

(correspondences between sorts and their sort predicates) Let  $w \in W$  with  $w = (i, n): S_k$  and let  $t \in I_w(s)$ . Let  $s = \tau$ . If  $t \in \mathcal{T}_{\tau,0}(w)$ , then by Definition 16 (1),  $p_\tau(t^n) \in S_k$ . Let  $s = \sigma$ . Since there is no ground anti-rigid sorted term,  $\mathcal{T}_{\sigma,0}(w) = \{a_n\}$ . If  $t = a_n$ , then  $p_\sigma(a_n^n) \in S_k$ . On the other hand, if  $t \in \mathcal{T}_{p_s,0}(w)$ , then by definition,  $p_{s'}(t^n) \in S_k$  with  $s' \leq s$ . When  $s' \neq s$ , by Definition 16 (2),  $p_s(t^n) \in S_k$ . Thus,  $I_w(s) \subseteq I_w(p_s)$ .

Let  $s$  be inextensible. If  $w \in W$  with  $w = (i, n): S_k$  and  $t \in I_w(p_s)$ , then  $p_s(t^n) \in S_k$ . By Definition 17 (6-a) and (6-d),  $t \in \mathcal{T}_{p_s,0}(w) (\subseteq I_w(s))$ . From this,  $I_w(p_s) \subseteq I_w(s)$  follows.

(Definition 13) 1. by Definition 17 (4), it is trivial. 2. Let  $w \in W$  and  $t \in U_w$ . Let  $\tau'$  be an inextensible type with  $p_{\tau'}: \tau \in \Omega$ . If  $t \notin I_w(\tau)$ , then by (v-5),  $t \notin I_w(p_{\tau'})$ . By the above correspondence,  $t \notin I_w(\tau')$ . For the other case, let  $t \in I_w(\tau')$ . If  $t \in \mathcal{T}_{\tau',0}(w)$ , then by Definition 16 (1),  $p_{\tau'}(t^n) \in S_k$  with  $w = (i, n): S_k$ . If  $t \in \mathcal{T}_{p_{\tau'},0}(w)$ , then by definition,  $p_s(t^n) \in S_k$  with  $s \leq \tau'$ . By Definition 16 (2),  $p_{\tau'}(t^n) \in S_k$ . On the other hand, the set  $(S^+)^1$  contains the following formula:

$$\blacksquare(\forall x_\tau)(p_{\tau'}(x_\tau) \rightarrow \blacksquare p_{\tau'}(x_\tau))$$

Hence, by Definition 17 (1-b), it is included in any world in  $W$ . By Definition 16 (8) and Lemma 1 (2),  $(\forall x_\tau)(p_{\tau'}(x_\tau) \rightarrow \blacksquare p_{\tau'}(x_\tau)) \in S_k$ . By Definition 16 (5),  $p_{\tau'}(t') \rightarrow \blacksquare p_{\tau'}(t') \in S_k$  for every ground term  $t' \in \mathcal{T}_{\tau,0}(w) \cup \mathcal{T}_{p_{\tau},0}(w) (= I_w(\tau))$ . Since  $p_{\tau'}(t^n) \in S_k$ , we have  $\blacksquare p_{\tau'}(t) \in S_k$ . Let  $(w, w') \in R'$ . Then, by Definition 17 (1-b),  $p_{\tau'}(t) \vee \neg \mathcal{E}(p_{\tau'}(t)) \in S'_k$  with  $w' = (i', n+m): S'_k$ . If  $t \in U_{w'}$  (iff  $\neg E(t) \notin S'_k$ ), then  $p_{\tau'}(t^{n+m})$  must belong to  $S'_k$ . Since  $p_{\tau'}$  is inextensible,  $t \in \mathcal{T}_{p_{\tau'},0}(w')$ . Therefore, by Definition 17 (6-a),  $t \in I_{w'}(\tau')$  ( $= \mathcal{T}_{\tau',0}(w') \cup \mathcal{T}_{p_{\tau'},0}(w')$ ).

Let  $w \in W$  and  $t \in U_w$ . Let  $\sigma$  be an anti-rigid sort with  $p_\sigma: \tau \in \Omega$ . If  $t \notin I_w(\tau)$ , then by (v-5),  $t \notin I_w(p_\sigma)$ . By the above correspondence,  $t \notin I_w(\sigma)$ . For the other case, let  $t \in I_w(\sigma)$  with  $w = (i, n): S_k$ . Since there is no ground anti-rigid sorted term,  $\mathcal{T}_{\sigma,0}(w) = \{a_n\}$ . If  $t = a_n$ , then  $p_\sigma(a_n^n) \in S_k$ . If  $t \in \mathcal{T}_{p_\sigma,0}(w)$ , then by definition,  $p_{\sigma'}(t^n) \in S_k$  with  $\sigma' \leq \sigma$ . When  $\sigma' \neq \sigma$ , by Definition 16 (2),  $p_\sigma(t^n) \in S_k$ . On the other hand, the set  $(S^+)^1$  contains the following formula:

$$\blacksquare(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \blacklozenge(\neg p_\sigma(x_\tau)))$$

By Definition 17 (1-b), every world in  $W$  contains the formula. According to Definition 16 (8) and Lemma 1 (2),  $(\forall x_\tau)(p_\sigma(x_\tau) \rightarrow \blacklozenge(\neg p_\sigma(x_\tau))) \in S_k$ . By Definition 16 (5) and (4),  $\neg p_\sigma(t') \in S_k$  or  $\blacklozenge(\neg p_\sigma(t')) \in S_k$  for every ground term  $t' \in \mathcal{T}_{\tau,0}(w) \cup \mathcal{T}_{p_\tau,0}(w) (= I_w(\tau))$ . Since  $p_\sigma(t^n) \in S_k$  and  $S_k$  is consistent,  $\blacklozenge(\neg p_\sigma(t)) \in S_k$ . By Definition 17 (1-b), there exists  $w' = (W, n+1): S'_k$  such that  $\neg p_\sigma(t), E(t^{n+1}) \in S'_k$ . So,  $p_\sigma(t^{n+1}) \notin S'_k$ , and by Definition 16 (2), for any  $\sigma'$  with  $\sigma' \leq \sigma$ ,  $p_{\sigma'}(t^{n+1}) \notin S'_k$ . Hence,  $t \notin \mathcal{T}_{p_\sigma,0}(w')$ . Since  $\mathcal{T}_{\sigma,0}(w') = \{a_{n+1}\}$  and  $t \neq a_{n+1}$ ,  $t \notin \mathcal{T}_{\sigma,0}(w')$ . Therefore, by Definition 17 (6-a),  $t \notin I_{w'}(\sigma)$  with  $t \in U_{w'}$ .

Let  $w_i, w_j \in W$  and let  $c_\tau$  or  $f_{\tau^*,\tau}(t_1, \dots, t_l)$  in  $U_{w_i} \cap U_{w_j}$ . Definition 17 (6-b) entails  $I_{w_i}(c) = c_\tau = I_{w_j}(c)$  and  $I_{w_i}(f)(t_1, \dots, t_l) = f_{\tau^*,\tau}(t_1, \dots, t_l) = I_{w_j}(f)(t_1, \dots, t_l)$  where  $c: \rightarrow \tau \in \Omega$  and  $f: \tau_1 \times \dots \times \tau_l \rightarrow \tau \in \Omega$ . 3. Similar to 2. The other conditions 4-6 in Definition 13 immediately follow from Definition 17 (2)-(4) and Lemma 6.

(ii) We next verify that for every  $w \in W$  with  $w = (i, n): S_k$  and for every  $F \in S_k$ ,  $w \models F'$  where  $F'$  is the non-annotated formula of  $F$ .

Let  $F = p(t_1^n, \dots, t_l^n)$ . Due to  $p(t_1^n, \dots, t_l^n) \in S_k$ , it is clear that  $w \models p(t_1, \dots, t_l)$  (by Definition 17 (6-d)).

Let  $F = \neg p(t_1, \dots, t_l)$ . Since  $(i, n): S_k$  is tab-consistent,  $p(t_1^n, \dots, t_l^n) \notin S_k$ . Thus,  $w \not\models p(t_1, \dots, t_l)$ .

Let  $F = F_1 \wedge F_2$ . Then,  $F_1^n, F_2^n \in S_k$  because  $w$  is saturated. By the induction hypothesis,  $w \models F_1$  and  $w \models F_2$ . So,  $w \models F_1 \wedge F_2$ .

Let  $F = F_1 \vee F_2$ . Then,  $F_1^n \in S_k$  or  $F_2^n \in S_k$  since  $w$  is saturated. By the induction hypothesis,  $w \models F_1$  or  $w \models F_2$ . Hence,  $w \models F_1 \vee F_2$ .

Let  $F = (\forall x_s)F_1$ . By Definition 16 (5),  $F_1[x_s/t]^n \in S_k$  for every ground term  $t \in \mathcal{T}_{s,0}(w) \cup \mathcal{T}_{p_s,0}(w)$  ( $= I_w(s)$ ). By the induction hypothesis,  $w \models F_1[x_s/t]$ . This yields  $w \models (\forall x_s)F_1$ .

Let  $F = (\exists x_s)F_1$ . By Definition 16 (6),  $p_s(c_\tau^n) \in S_k$  and  $F_1[x_s/c_\tau]^n \in S_k$  for some constant  $c_\tau$ . If  $s \in T$ , then  $s = \tau$ . Thus,  $c_\tau \in \mathcal{T}_0 \cap \mathcal{T}_\tau$  and it occurs in  $S_k$ . This implies  $c_\tau \in \mathcal{T}_{s,0}(w)$  ( $\subseteq I_w(s)$ ). If  $s \notin T$ , then  $p_s$  must be inextensible (since  $s$  is an anti-rigid sort). Then,  $c_\tau \in \mathcal{T}_{p_s,0}(w)$  ( $\subseteq I_w(s)$ ). By the induction hypothesis,  $w \models F_1[x_s/c_\tau]$ . Hence,  $w \models (\exists x_s)F_1$ .

Let  $F = \Box_j F_1$ . Let  $w' \in W$  such that  $(w, w') \in R_j$  and  $w' = (j, n+m): S'_k$ . By Definition 17 (1-b)-i and ii,  $F_1 \vee \neg \mathcal{E}(F_1) \in S'_k$ . By Definition 16 (4),  $F_1^{n+m} \in S'_k$  or  $\neg \mathcal{E}(F_1) \in S'_k$  (if  $\mathcal{E}(F_1) \neq \emptyset$ ). By the induction hypothesis, if  $F \notin Nex_{w'}$ , then  $w' \models F_1$ . Therefore,  $w \models \Box_j F_1$ .

Let  $F = \Diamond_j F_1$ . By Definition 17 (1-b)-i and ii, there exists  $w' = (j, n+1): S'_k$  such that  $(w, w') \in R_j$  and  $F_1 \wedge \mathcal{E}(F_1) \in S'_k$ . Since  $S'_k$  is saturated,  $F_1 \in S'_k$  and  $\mathcal{E}(F_1) \in S'_k$  (if  $\mathcal{E}(F_1) \neq \emptyset$ ). By the induction hypothesis,  $w' \models F_1$  and by Lemma 1 (1),  $F_1 \notin Nex_{w'}$ . By definition,  $w \models \Diamond_j F_1$ .

Let  $F = \blacksquare F_1$ . Let  $w' \in W$  such that  $(w, w') \in R'$  and  $w' = (i', n+m): S'_k$ . By Definition 17 (1-b)-iii,  $F_1 \vee \neg \mathcal{E}(F_1) \in S'_k$ . By Definition 16 (4),  $F_1^{n+m} \in S'_k$  or  $\neg \mathcal{E}(F_1) \in S'_k$  (if  $\mathcal{E}(F_1) \neq \emptyset$ ). By the induction hypothesis, if  $F \notin Nex_{w'}$ , then  $w' \models F_1$ . Hence,  $w \models \blacksquare F_1$ .

Let  $F = \blacklozenge F_1$ . By Definition 17 (1-b)-iii, there exists  $w' = (W, n+1): S'_k$  such that  $(w, w') \in R'$  and  $F_1 \wedge \mathcal{E}(F_1) \in S'_k$ . Since  $S'_k$  is saturated,  $F_1 \in S'_k$  and  $\mathcal{E}(F_1) \in S'_k$  (if  $\mathcal{E}(F_1) \neq \emptyset$ ). By the induction hypothesis,  $w' \models F_1$  and by Lemma 1 (1),  $F_1 \notin Nex_{w'}$ . Therefore,  $w \models \blacklozenge F_1$ . ■