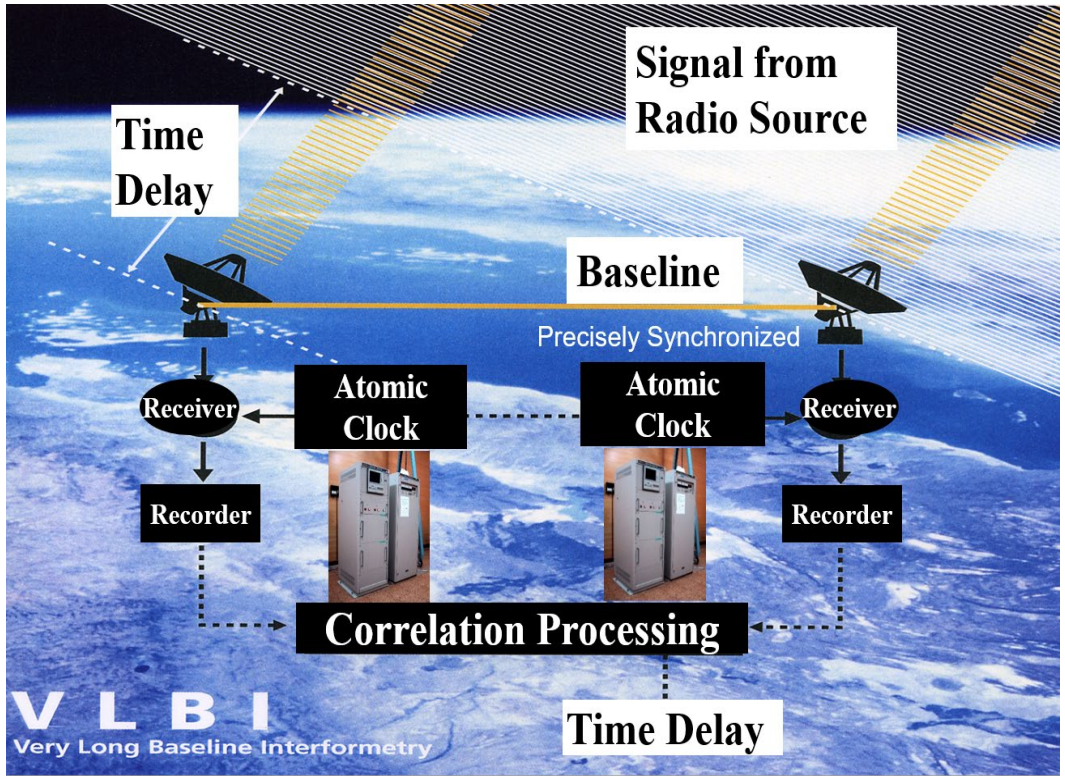
Time System and General relativity

Time system on the Geoid : TT, TAI, UTC
Geocentric Coordinate Time :TCG
Barycentric Coordinate Time :TCB, TDB

National Institute of Information and Communications Technology
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Mamoru SEKIDO

VLBI Observation Equation based on General relativity

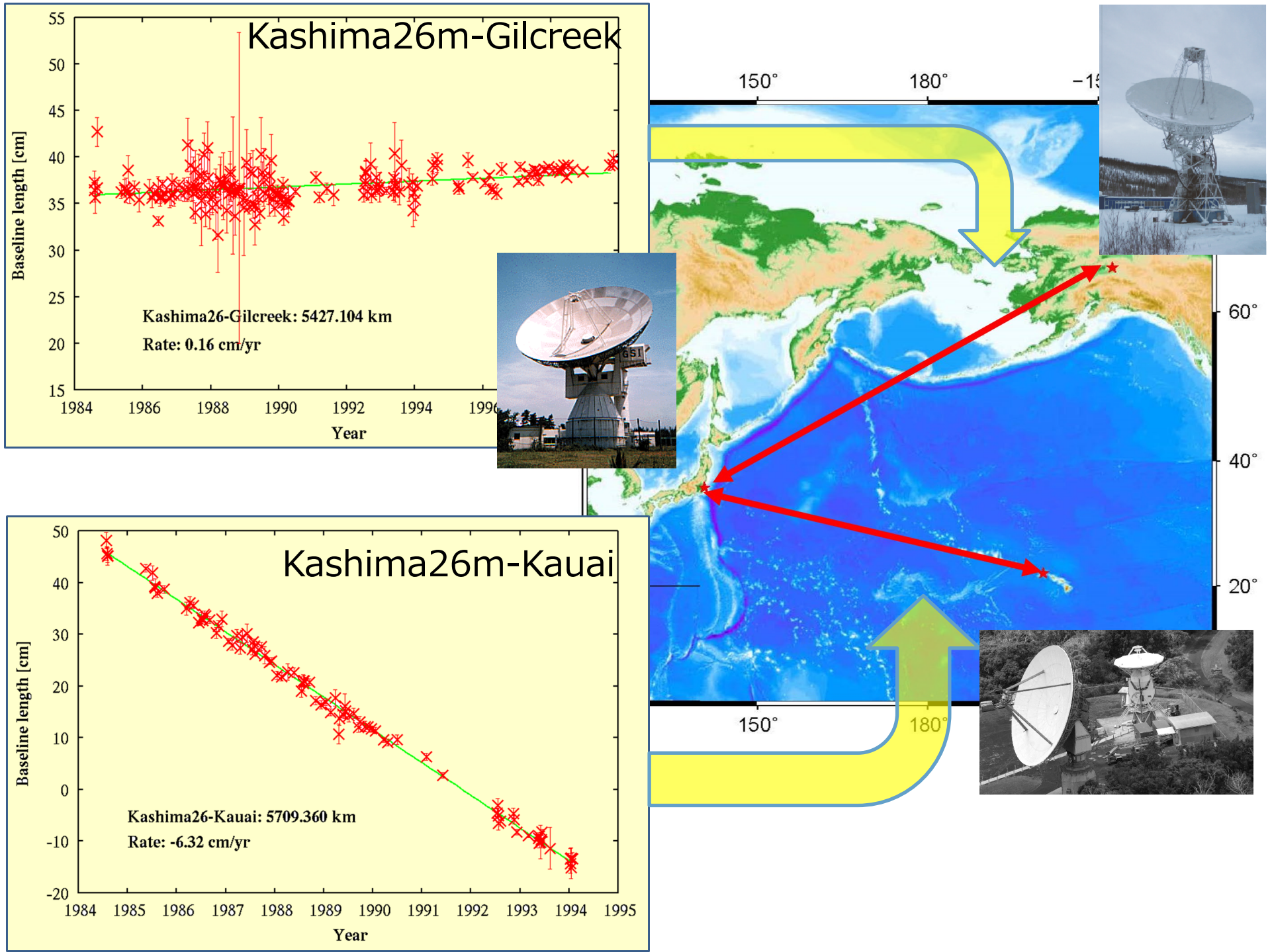
1. Signal arrival time difference (time delay) between pair of radio telescopes is observed.
2. Physical parameters (Baseline vector, Source structure, etc..) are estimated from the delay data for radio sources in different directions.
3. Delay model is inherently non-linear w.r.t. the parameters. Linearized equation for small quantity (O-C) is solved with accurate prediction model.



$$\tau = -\frac{\vec{S}\cdot\vec{B}}{c} + \tau_{\mathrm{clk}} + \tau_{\mathrm{atm}} \cdots$$

$$\tau = -\frac{\left\{-\left[1 - 2\frac{w_E}{c^2} - \frac{|\boldsymbol{V}_E|^2 + 2\boldsymbol{V}_E\cdot\boldsymbol{v}_2}{2c^2}\right]\frac{\boldsymbol{s}\cdot\boldsymbol{b}}{c} - \frac{\boldsymbol{V}_E\cdot\boldsymbol{b}}{c^2}\left(1 + \frac{\boldsymbol{s}\cdot\boldsymbol{V}_E}{2c}\right) + \Delta\tau_g\right\}}{1 + \boldsymbol{s}\cdot\frac{\boldsymbol{V}_E + \boldsymbol{v}_2}{\boldsymbol{c}}}$$

$$+\tau_{\mathrm{clk}} + \tau_{\mathrm{atm}} + \tau_{ion} + \tau_{str} + \Delta\tau_g$$


Kashima26m-Gilcreek
Baseline length [cm]
Year
Kashima26-Gilcreek: 5427.104 km
Rate: 0.16 cm/yr
Kashima26m-Kauai
Baseline length [cm]
Year
Kashima26-Kauai: 5709.360 km
Rate: -6.32 cm/yr
150°
180°
60°
40°
20°

Magnitude of relativistic effect ?

VLBI Observable and its precision

- Baseline Length B=6400km (Earth Radius) -> 21.3ms ($\tau \sim 2.13 \times 10^{-2}$ [s])
- Meas. Precision: -> 1ps~0.1ps ($\delta\tau \sim 10^{-12} \sim 10^{-13}$ [s])

Relativistic Effect

- Velocity: $\times \left(\frac{v}{c}\right)^2$
  - Earth rotation@Equator :465 $m/s \Rightarrow \left(\frac{v}{c}\right)^2 = \mathbf{2.4 \times 10^{-12}}$, $\tau \cdot \left(\frac{v}{c}\right)^2 = 5 \times 10^{-14}$ [s] (=0.05 [ps])
  - Earth evolution :3×10^4 m/s $\Rightarrow \left(\frac{v}{c}\right)^2 = \mathbf{1 \times 10^{-8}}$, $\tau \cdot \left(\frac{v}{c}\right)^2 = 2 \times 10^{-10}$ [s] (=211 [ps])
- Gravitational Potential: $\times \frac{U}{c^2}$、 $U = \frac{GM}{r}$
  - Earth（Earth Surface) : $U/c^2 \sim \mathbf{7 \times 10^{-10}}$, $\tau \cdot U/c^2 = 1.5 \times 10^{-11}$ [s] (=15 [ps])
  - Sun（Earth Orbit) : $U/c^2 \sim \mathbf{1 \times 10^{-8}}$, $\tau \cdot U/c^2 = 2.1 \times 10^{-10}$ [s] (=210 [ps])
  - Jupiter (Jipiter Orbit) : $U/c^2 \sim 2 \times 10^{-12}$, $\tau \cdot U/c^2 = 4 \times 10^{-14}$ [s] (=0.04 [ps])

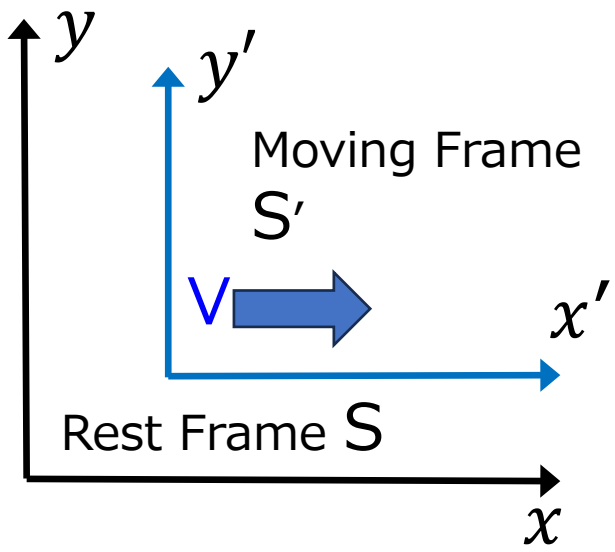
Lorentz transformation

Simple 1-D case

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \frac{1}{\sqrt{1-V^2/c^2}} \begin{pmatrix} 1 & V/c \\ V/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma_s \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

$$\gamma_s \equiv \frac{1}{\sqrt{1-V^2/c^2}}, \quad \beta \equiv \frac{V}{c}$$

$$V \equiv \sqrt{v_x^2 + v_y^2 + v_z^2}$$



Lorentz contraction

A length rest w.r.t. S′ frame is measured from S frame. Length measurement must be made simultaneously in S frame (dt=0).

$$\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \gamma_s \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 0 \\ dx \end{pmatrix} = \gamma_s \begin{pmatrix} -\beta dx \\ dx \end{pmatrix} \Rightarrow dx = \frac{dx'}{\gamma_s} < dx'$$

A length in moving system is measured shorter in rest frame.

This is due to difference of simultaneity between mutually moving frames.

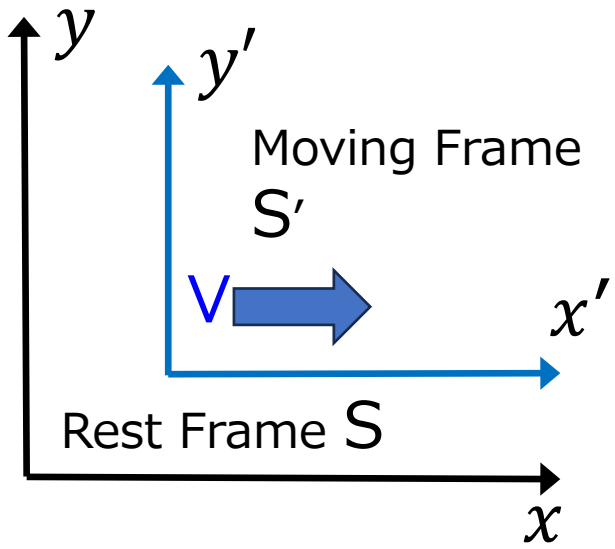
Lorentz transformation

Simple 1-D case

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \frac{1}{\sqrt{1-V^2/c^2}} \begin{pmatrix} 1 & V/c \\ V/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma_s \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

$$\gamma_s \equiv \frac{1}{\sqrt{1-V^2/c^2}}, \quad \beta \equiv \frac{V}{c}$$

$$V \equiv \sqrt{v_x^2 + v_y^2 + v_z^2}$$



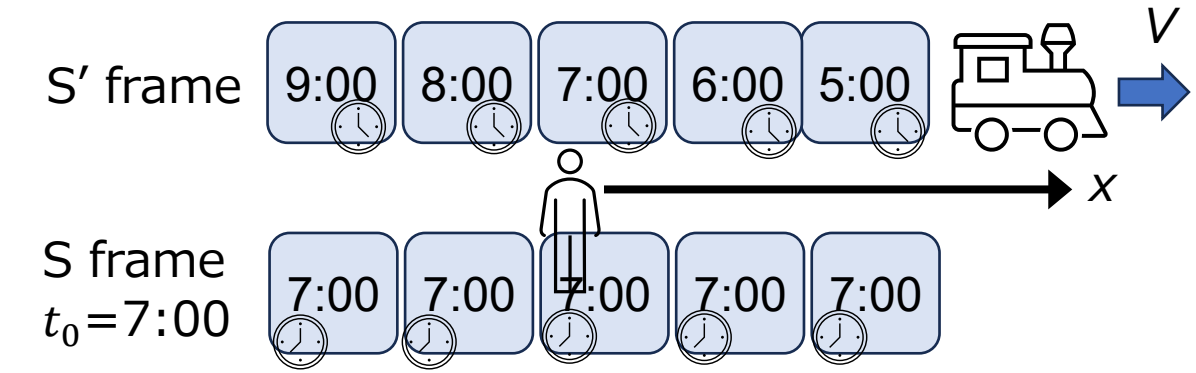
Difference of Simultaneity

Simultaneous event in S (rest frame) is not simultaneous in S′(moving frame)

$$\begin{pmatrix} 0 \\ dx \end{pmatrix} = \gamma_s \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \gamma_s \begin{pmatrix} cdt' + \beta dx' \\ dx' + Vdt' \end{pmatrix},$$

$$dt' = -\frac{Vdx'}{c^2}$$

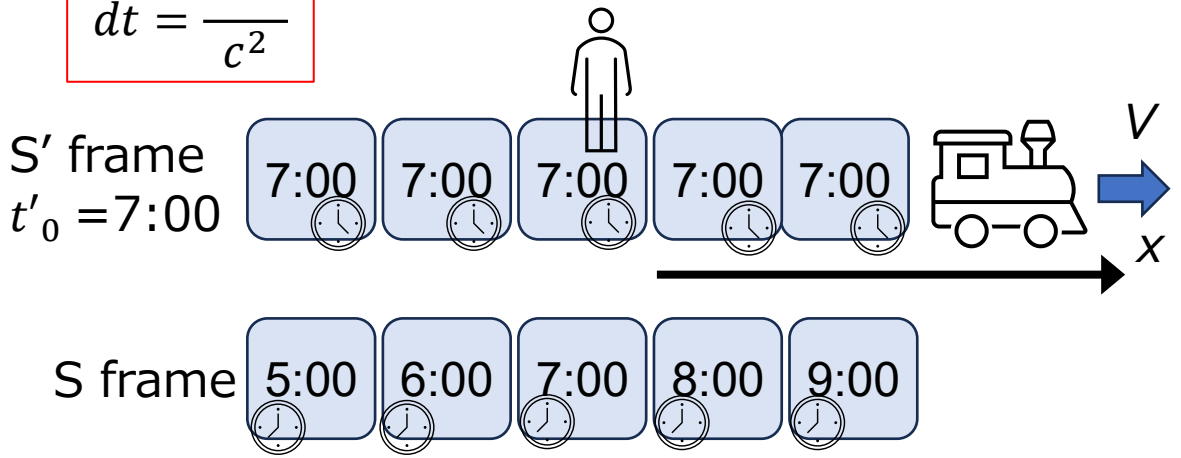
This means…



Inversely, a moment simultaneous in S' frame (dt'=0) corresponds to different epochs depending on the position in S frame.

$$\begin{pmatrix} 0 \\ dx' \end{pmatrix} = \gamma_s \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix} = \gamma_s \begin{pmatrix} cdt - \beta dx \\ dx - Vdt \end{pmatrix}$$

$$dt = \frac{Vdx}{c^2}$$



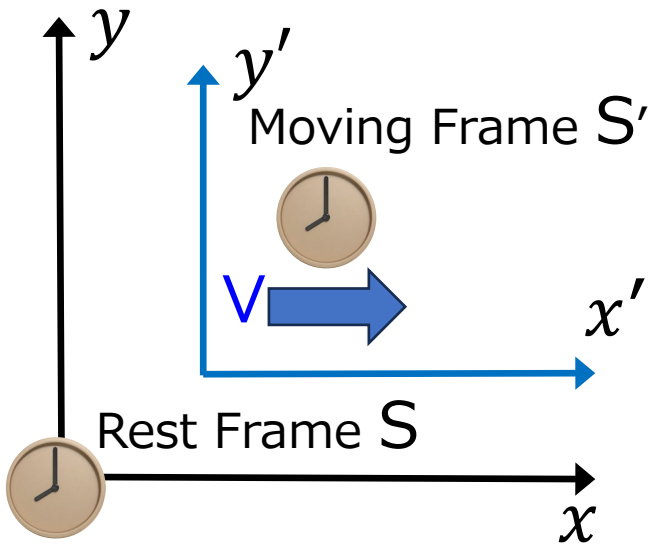
Lorentz transformation

Simple 1-D case

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \frac{1}{\sqrt{1-V^2/c^2}} \begin{pmatrix} 1 & V/c \\ V/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma_s \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

$$\gamma_s \equiv \frac{1}{\sqrt{1-V^2/c^2}}, \quad \beta \equiv \frac{V}{c}$$

$$V \equiv \sqrt{v_x^2 + v_y^2 + v_z^2}$$



Time dilation

A time interval dt’ in S’ frame is shorter than that in S frame.

$$\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \gamma_s \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix} = \gamma_s \begin{pmatrix} cdt - \beta dx \\ dx - c\beta dt \end{pmatrix} = \gamma_s \begin{pmatrix} cdt(1-\beta^2) \\ 0 \end{pmatrix} = \begin{pmatrix} c\,dt/\gamma_s \\ 0 \end{pmatrix} \Rightarrow dt' = \frac{dt}{\gamma_s} < dt$$

Time in S’ passes slower than S, in other words.

Lorentz transformation

$$\gamma_s \equiv \frac{1}{\sqrt{1-V^2/c^2}}, \quad \beta \equiv \frac{V}{c}$$
$$V \equiv \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The simplest 1-D case

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \frac{1}{\sqrt{1-V^2/c^2}} \begin{pmatrix} 1 & V/c \\ V/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma_s \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

Lorentz transformation in 4-D

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma_s & \gamma_s \frac{v_x}{c} & \gamma_s \frac{v_y}{c} & \gamma_s \frac{v_z}{c} \\ \gamma_s \frac{v_x}{c} & 1 + (\gamma_s - 1)\frac{v_x^2}{V^2} & (\gamma_s - 1)\frac{v_x v_y}{V^2} & (\gamma_s - 1)\frac{v_x v_z}{V^2} \\ \gamma_s \frac{v_y}{c} & (\gamma_s - 1)\frac{v_y v_x}{V^2} & 1 + (\gamma_s - 1)\frac{v_y^2}{V^2} & (\gamma_s - 1)\frac{v_y v_z}{V^2} \\ \gamma_s \frac{v_z}{c} & (\gamma_s - 1)\frac{v_z v_x}{V^2} & (\gamma_s - 1)\frac{v_z v_y}{V^2} & 1 + (\gamma_s - 1)\frac{v_z^2}{V^2} \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

vector expression is given as

$$= \begin{pmatrix} \gamma_s & \gamma_s \frac{\overrightarrow{\boldsymbol{v}^t}}{c} \\ \gamma_s \frac{\vec{\boldsymbol{v}}}{c} & \boldsymbol{I} + (\gamma_s - 1)\frac{\vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{v}}}{V^2} \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

Where,

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \left(\vec{\boldsymbol{a}} \otimes \vec{\boldsymbol{b}}\right)_{i,j} = a_i b_j$$

Theory of relativity tells that space-time is 4-D geometry.

Invariant quantity w.r.t. coordinate transformation:
infinitesimal 4-D event distance (line element) ds is given with metric tensor $g_{\mu\nu}$ as :

$$(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu = \begin{pmatrix} dx^0 & dx^1 & dx^2 & dx^3 \end{pmatrix} \begin{pmatrix} g_{00} & \cdots & g_{03} \\ \vdots & \ddots & \vdots \\ g_{30} & \cdots & g_{33} \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

Einstein summation convention

$$g_{\mu\nu}dx^\mu dx^\nu = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu$$

where Greek character ($\alpha. \mu, \nu, \ldots$) runs 0 -- 3,
and roman character (I,j,k,…) runs over 1 -- 3

Flat (Mincowski) space case:

$$(ds)^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -(c\,dt)^2 + dx^2 + dy^2 + dz^2, \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This event distance is invariant w.r.t. Lorentz transformation. It is confirmed as follows in 2-D case.

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \underbrace{\gamma_s \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}}_{\boldsymbol{L}} \begin{pmatrix} ct' \\ x' \end{pmatrix}, \quad \eta_{\mu\nu} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{M}}$$

Lorentz transformation is expressed as $\boldsymbol{x = Lx'}$, and Mincowski metric tensor $\boldsymbol{M}$ gives $ds^2 = \boldsymbol{x}^t\boldsymbol{M}\boldsymbol{x} = \boldsymbol{x}'^t\boldsymbol{L}^t\boldsymbol{M}\boldsymbol{L}\boldsymbol{x}'$

$$\boldsymbol{L}^t\boldsymbol{M}\boldsymbol{L} = \gamma_s^2 \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} = \frac{1}{1-\beta^2} \begin{pmatrix} -(1-\beta^2) & 0 \\ 0 & 1-\beta^2 \end{pmatrix} = \boldsymbol{M}$$

Therefore, $\boldsymbol{x}^t\boldsymbol{M}\boldsymbol{x} = \boldsymbol{x}'^t\boldsymbol{M}\boldsymbol{x}' = ds^2$
is invariant w.r.t. Lorentz transformation.

A general space-time invariant event distance $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$

Post-Galilei metric tensor $g_{\mu\nu}$ is given as $g_{00} = -1 + \frac{2\phi}{c^2}$, $g_{0j} = g_{j0} = 0$, $g_{ij} = \delta_{ij} + \frac{2\phi}{c^2}\delta_{ij}$

Invariant event line element is

$$(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2\phi}{c^2}\right)c^2dt^2 + \left(1 + \frac{2\phi}{c^2}\right)(dx^2 + dy^2 + dz^2) \quad (1)$$

Equation of proper time

Coordinate time t: time coordinate of the reference frame

Proper time τ: time system co-moving with the object of concern (dx^μ=0)

$(ds)^2 = -c^2(d\tau)^2 = -\left(1 - \frac{2\phi}{c^2}\right)c^2dt^2 + \left(1 + \frac{2\gamma\phi}{c^2}\right)(dx^2 + dy^2 + dz^2)$. Dividing equation with $(dt)^2$ gives

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - \frac{2\phi}{c^2}\right) - \left(1 + \frac{2\phi}{c^2}\right)\left(\frac{V}{c}\right)^2 \approx \left(1 - \frac{2\phi}{c^2}\right) - \left(\frac{V}{c}\right)^2$$

Where $V = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

$$\frac{d\tau}{dt} = \sqrt{\left(1 - \frac{2\phi}{c^2}\right) - \left(\frac{V}{c}\right)^2} \approx 1 - \frac{\phi}{c^2} - \frac{1}{2}\left(\frac{V}{c}\right)^2$$

Approximation with Taylor expansion and retaining terms in order of $(V/c)^2$
$\sqrt{1+x} = 1 + \frac{1}{2}x^2 - \frac{1}{4}x^3 + \cdots$, and $V/c \sim 10^{-4}$, $\phi/c^2c \sim 10^{-8}$.

Proper time (clock co-moving with the object) rate is slower than coordinate time due to gravitational potential and velocity of the object.

Reference frame and Coordinate time

Reference frame

BCRS: Barycentric Celestial Reference System.
Its coordinate time is Barycentric Coordinate Time (**TCB**), or Barycentric Dynamical Time (**TDB**)

GCRS: Geocentric Celestial Refence System
Its coordinate time is Geocentric Coordinate Time (**TCG**), or Terrestrial Time (**TT**) as scaled time scale to be the same clock rate with .

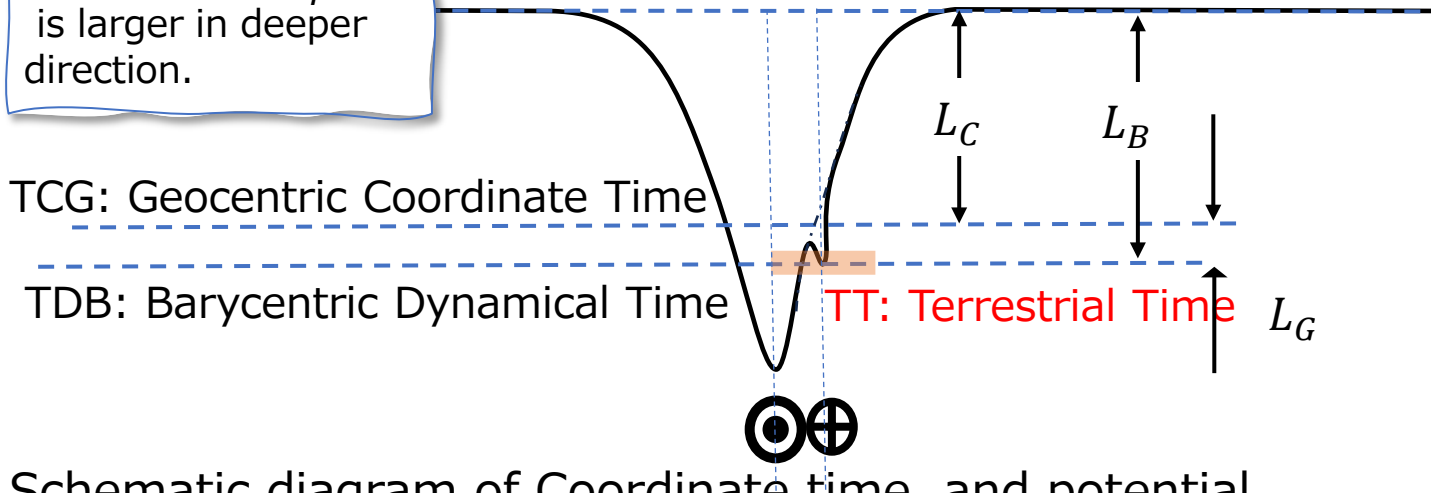
Time system related with TT

TT(Terrestrial Time): a coordinate time of scaled TCG to be the same rate with proper time of clock on the Geoid. TT=TAI+32.184 s

TAI (Temps Atomique International): Time system generated as weighted average of primary frequency standards in the world. It is regarded as realization of TT.

UTC (Coordinated Universal Time):Time system with additional offset to TAI to be coincide with TAI within 0.9 s.

Gravitational potential $U = \frac{GM}{r}$ is larger in deeper direction.



Schematic diagram of Coordinate time and potential

$$\frac{d\tau}{dt} = 1 - \frac{U}{c^2} - \frac{1}{2}\left(\frac{V}{c}\right)^2 \Rightarrow 1 - \left\langle \frac{U}{c^2} + \frac{1}{2}\left(\frac{V}{c}\right)^2 \right\rangle$$

$$= 1 - L_B \ : (\mathrm{TCB} \leftrightarrow \mathrm{TDB})$$

$$= 1 - L_G \ : (\mathrm{TCG} \leftrightarrow \mathrm{TT})$$

$$L_B = 1 - \frac{d(TDB)}{d(TCB)} = 1.550519768 \times 10^{-8}$$

$$L_G = 1 - \frac{d(TT)}{d(TCG)} = 6.969290134 \times 10^{-10}$$

References

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  - Relativistic Astrometry and AstroDynamics: Lecture note https://www2.nict.go.jp/sts/stmg/vcon/WG/VLBI-Book/References/Fukushima-note.pdf
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- T.D.Moyer, Radiometric Tracking Techniques for Deep-Space Navigation, https://descanso.jpl.nasa.gov/monograph/series2_section.html
- IERS Conventions(2010) https://www.iers.org/IERS/EN/DataProducts/Conventions/conventions.html